(1a) Prove $\begin{cases} f:\mathbb{C}\to\mathbb{C} \\ f(z)=3z-7 \end{cases}$ is continuous.

Proof: Let $\epsilon>0$ and choose $\delta=\frac{\epsilon}{3}$. If $|z-z_0|<\delta$, then

$$|f(z)-f(z_0)|=|(3z-7)-(3z_0-7)|$$

$$=3|z-z_0|$$

$$<3\delta=3\left(\frac{\epsilon}{3}\right)=\epsilon. \quad \square$$

(1b) Prove $\begin{cases} f:\mathbb{C}\to\mathbb{C} \\ f(z)=7 \end{cases}$ is continuous.

Proof: Let $\epsilon>0$ and choose any $\delta>0$. If $|z-z_0|<\delta$, then

$$|f(z)-f(z_0)|=|7-7|=0<\epsilon. \quad \square$$

(1c) Prove $\begin{cases} f:\mathbb{C}\to\mathbb{C} \\ f(z)=\mathrm{Re}(z) \end{cases}$ is continuous.

Proof: Let $\epsilon>0$ and choose $\delta=\epsilon$.

Note that

$$|z-z_0|=|(x+iy)-(x_0+iy_0)|$$

$$=|(x-x_0)+(y-y_0)i|$$

$$=\sqrt{(x-x_0)^2+(y-y_0)^2} \leftarrow \text{positive}$$

$$\overset{(*)}{\geq}\sqrt{(x-x_0)^2}$$

$$=|x-x_0|.$$

If $|z-z_0|<\delta$, then by $(*)$,

$$|f(z)-f(z_0)|=|\mathrm{Re}(z)-\mathrm{Re}(z_0)|=|x-x_0|\overset{(*)}{\leq}|z-z_0|<\delta=\epsilon. \quad \square$$

2a $f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$

$= \lim_{h\to 0} \frac{1}{h}\left[(3(z+h)^2+1)-(3z^2+1)\right]$

$= \lim_{h\to 0} \frac{1}{h}\left[\cancel{3z^2}+6zh+3h^2\cancel{+1}\cancel{-3z^2}\cancel{-1}\right]$

$= \lim_{h\to 0} 6z+3h$

$= 6z$

2b $f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$

$= \lim_{h\to 0} \frac{1}{h}\left[\{(z+h)^3-5(z+h)\}-\{z^3-5z\}\right]$

$= \lim_{h\to 0} \frac{1}{h}\left[\cancel{z^3}+6z^2h+3zh^2+h^3\cancel{-5z}-5h\cancel{-z^3}\cancel{+5z}\right]$

$= \lim_{h\to 0} 6z^2+3zh+h^2-5$

$= 6z^2-5$

3a $f(x+iy) = 3(x+iy)^2+1 = 3[x^2+2xyi+i^2y^2]+1$

$= \underbrace{(3x^2-3y^2+1)}_{u(x,y)} + \underbrace{6xy}_{v(x,y)}\, i$

$\frac{\partial u}{\partial x} = 6x \quad \frac{\partial v}{\partial x} = 6y$

$\frac{\partial u}{\partial y} = -6y \quad \frac{\partial v}{\partial y} = 6x$

$\Longrightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ ✓ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ ✓

3c $f(x+iy) = \frac{1}{x+iy} = \left(\frac{1}{x+iy}\right)\left(\frac{x-iy}{x-iy}\right) = \frac{x-iy}{x^2+y^2} = \underbrace{\frac{x}{x^2+y^2}}_{u(x,y)} - \underbrace{\frac{y}{x^2+y^2}}_{v(x,y)}\, i$

$\frac{\partial u}{\partial x} = \frac{(x^2+y^2)(1)-x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{x^2+y^2}$

$\frac{\partial v}{\partial x} = -\left[\frac{0-y(2x)}{(x^2+y^2)^2}\right] = \frac{2xy}{(x^2+y^2)^2}$

$\frac{\partial u}{\partial y} = \frac{(x^2+y^2)(0)-x(2y)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$

$\frac{\partial v}{\partial y} = -\left[\frac{(x^2+y^2)(1)-y(2y)}{(x^2+y^2)^2}\right] = -\left[\frac{x^2-y^2}{x^2+y^2}\right] = \frac{y^2-x^2}{x^2+y^2}$

$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ ✓

(3d) $f(x+iy) = e^x e^{iy} = \underbrace{e^x\cos(y)}_{u} + \underbrace{e^x\sin(y)}_{v}\, i$

$$\left.\begin{matrix} \dfrac{\partial u}{\partial x} = e^x\cos(y) & \dfrac{\partial u}{\partial y} = -e^x\sin(y) \\ \dfrac{\partial v}{\partial x} = e^x\sin(y) & \dfrac{\partial v}{\partial y} = e^x\cos y \end{matrix}\right\} \Rightarrow \begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \checkmark \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \checkmark \end{cases}$$

(4a) $f(x+iy) = \overbrace{(x+iy)}^{z} + \overbrace{(x-iy)}^{\bar{z}} = \underset{u}{2x} + \underset{v}{0y}$

$$\begin{matrix} \dfrac{\partial u}{\partial x} = 2, & \dfrac{\partial v}{\partial y} = 0 \\ \dfrac{\partial u}{\partial y} = 0, & \dfrac{\partial v}{\partial x} = 0 \end{matrix} \Rightarrow \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \Rightarrow \text{not } \mathbb{C}\text{-diff'bl}$$

(4b) $f(x+iy) = e^x e^{-iy} = e^x\cos(-y) + e^x\sin(-y)\, i$

$\boxed{\text{cos is even} \\ \text{sin is odd}} \rightarrow = \underbrace{e^x\cos(y)}_{u} - \underbrace{e^x\sin(y)}_{v}\, i$

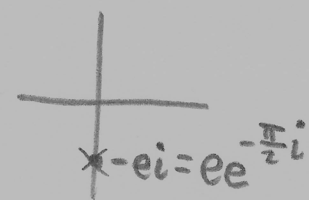
$$\left.\begin{matrix} \dfrac{\partial u}{\partial x} = e^x\cos(y) & \dfrac{\partial u}{\partial y} = -e^x\sin(y) \\ \dfrac{\partial v}{\partial x} = -e^x\sin(y) & \dfrac{\partial v}{\partial y} = -e^x\cos(y) \end{matrix}\right\} \Rightarrow \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$$

$\Rightarrow$ not $\mathbb{C}$ diffb'l

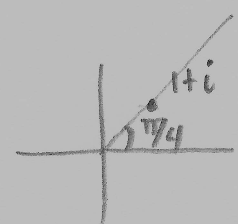
5a) $$e^{-7+9\pi i} = e^{-7}\left[\cos(9\pi) + i\sin(9\pi)\right]$$
$$= e^{-7}\left[-1+0\right]$$
$$= -e^{-7}$$

5b) $$e^{-\frac{17}{4}+\frac{\pi}{4}i} = e^{-\frac{17}{4}}e^{\frac{\pi}{4}i} = e^{-\frac{17}{4}}\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right]$$
$$= e^{-\frac{17}{4}}\frac{\sqrt{2}}{2} + e^{-\frac{17}{4}}\frac{\sqrt{2}}{2}i$$

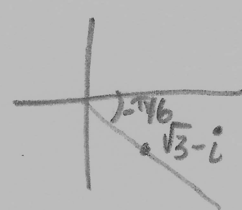
5c) $$\mathrm{Log}(-ei) = \ln(|-ei|) + i\,\mathrm{Arg}(-ei)$$
$$= \ln(e) + i\left(-\frac{\pi}{2}\right)$$
$$= 1 - \frac{\pi}{2}i$$



5d) $$\mathrm{Log}(1+i) = \ln(|1+i|) + i\,\mathrm{Arg}(1+i)$$
$$= \ln(\sqrt{2}) + \frac{\pi}{4}i$$



5e) $$\log(\sqrt{3}-i) = \ln(|\sqrt{3}-i|) + \arg(\sqrt{3}-i)i$$
$$= \ln\left(\sqrt{3+1}\right) + \left(-\frac{\pi}{6} + 2n\pi\right)i$$
$$= \ln(2) + \left(-\frac{\pi}{6} + 2n\pi\right)i$$



5f) $$\log(1-i) = \ln(|1-i|) + i\arg(1-i)$$
$$= \ln(\sqrt{2}) + \left(-\frac{\pi}{4} + 2n\pi\right)i$$