Homework 3 MATH 1199 Fall 2019



1a) Prove $\lim_{z\to 5} z+1 = 6$

Proof: Let $\epsilon > 0$ and choose $\delta = \epsilon$. If $|z-5| < \delta$, then

$$|(z+1)-6| = |z-5| < \delta = \epsilon,$$

completing the proof. ■

1b) Prove $\lim_{z\to 3i} \frac{z+1}{2} = \frac{3i+1}{2}$.

Proof: Let $\epsilon > 0$ choose $\delta = 2\epsilon$. If $|z-3i| < \delta$, then

$$\left|\frac{z+1}{2} - \frac{3i+1}{2}\right| = \frac{|z-3i|}{2} < \frac{2\epsilon}{2} = \epsilon,$$

completing the proof. ■

1c) Prove $\lim_{z\to \frac{1+i}{5}} \frac{5z-i}{7} = \frac{1}{7}$.

Proof: Let $\epsilon > 0$ and choose $\delta = \frac{7\epsilon}{5}$. If $\left|z - \frac{1+i}{5}\right| < \delta$, then

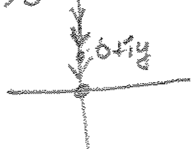
$$\left|\frac{5z-i}{7} - \frac{1}{7}\right| = \frac{1}{7}\left|5z - i - 1\right|$$

$$= \frac{5}{7}\left|z - \frac{i+1}{5}\right|$$

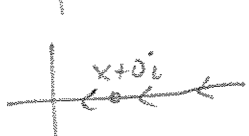
$$< \cancel{\frac{5}{7}}\,\cancel{\frac{7\epsilon}{5}} = \epsilon,$$

completing the proof. ■

2) Show $\lim_{z\to 0} \frac{z}{|z|}$ DNE.

Path 1: 


$$\lim_{y\to 0} \frac{0+iy}{|0+iy|} = \lim_{y\to 0} \frac{iy}{y} = i$$

Path 2: 


$$\lim_{x\to 0} \frac{x+0i}{|x+0i|} = \lim_{x\to 0} \frac{x}{x} = 1$$

The limits disagree, so $\lim_{z\to 0} \frac{z}{|z|}$ DNE.

3a $\lim_{z\to\infty} \frac{1}{z} = 0$ means $\lim_{z\to 0} \underbrace{\frac{1}{\frac{1}{z}}}_{=z} = 0$

So, Prove $\lim_{z\to 0} z = 0$.

<u>Proof</u>: Let $\epsilon > 0$ choose $\delta = \epsilon$. If $|z-0| < \delta$, then $|z-0| < \delta = \epsilon$, completing the proof. ∎

3b $\lim_{z\to\infty} \frac{4z^2}{(z-1)^2} = 4$ means $\lim_{z\to 0} \underbrace{\frac{4\left(\frac{1}{z}\right)^2}{\left(\frac{1}{z}-1\right)^2}}_{=\frac{4}{(1-z)^2}} = 4$

So, need to prove $\lim_{z\to 0} \frac{4}{(1-z)^2} = 4$.

To do it, can first prove $\lim_{z\to 0} 1-z = 1$.

<u>Proof</u>: Let $\epsilon > 0$ and choose $\delta = \epsilon$. If $\overbrace{|z-0|}^{|z|} < \delta$, then $|(1-z)-1| = |z| < \delta = \epsilon$, completing the proof.

The theorem on p.24 shows us

$$\lim_{z\to 0} (1-z)^2 = \left[\lim_{z\to 0}(1-z)\right]\left[\lim_{z\to 0}(1-z)\right] = 1\cdot 1 = \underline{1}$$

Now prove $\lim_{z\to 0} 4 = 4$.

<u>Proof</u>: Let $\epsilon > 0$ and choose $\delta = \epsilon$. If $|z-0| < \delta$, then $|4-4| = 0 < \delta = \epsilon$, completing the proof. ∎

"obviously true"

Therefore by the theorem on p.24,

$$\lim_{z\to 0} \frac{4}{(1-z)^2} = \frac{\lim_{z\to 0} 4}{\lim_{z\to 0}(1-z)^2} = \frac{4}{1} = 4.$$

(3c) $\lim_{z\to\infty} \frac{z^2+1}{z-1} = \infty$ means $\lim_{z\to 0} \frac{1}{\underbrace{\left(\frac{\left(\frac{1}{z}\right)^2+1}{\frac{1}{z}-1}\right)}} = 0$

$$= \frac{\frac{1}{z}-1}{\left(\frac{1}{z}\right)^2+1} = \frac{z-z^2}{1+z^2}$$

<u>Claim</u>: $\lim_{z\to 0} \pm z = 0$

<u>Proof</u>: Let $\epsilon > 0$ choose $\delta = \epsilon$. If $|z-0| = |z| < \delta$, then
$|\pm z - 0| = |z| < \delta = \epsilon$. ∎

<u>Claim</u>: $\lim_{z\to 0} \pm 1 = \pm 1$

<u>Proof</u>: Let $\epsilon > 0$ choose $\delta = \epsilon$. If $|z-0| < \delta$, then
$|\pm 1 - \pm 1| = 0 < \delta = \epsilon$. ∎

<u>Claim</u>: $\lim_{z\to 0} \pm z^2 = 0$

<u>Pf</u>: $\lim_{z\to 0} \pm z^2 = \left(\lim_{z\to 0} \pm z\right)\left(\lim_{z\to 0} \pm z\right) = 0\cdot 0 = 0$

<u>Claim</u>: $\lim_{z\to 0} z - z^2 = 0$

<u>Pf</u>: $\lim_{z\to 0} z - z^2 = \left(\lim_{z\to 0} z\right) + \left(\lim_{z\to 0} -z^2\right) = 0 + 0 = 0$

<u>Claim</u>: $\lim_{z\to 0} 1 + z^2 = 1$

<u>Pf</u>: $\lim_{z\to 0} 1 + z^2 = \left(\lim_{z\to 0} 1\right) + \left(\lim_{z\to 0} z^2\right) = 1 + 0 = 1$

Finally,

$$\lim_{z\to 0} \frac{z-z^2}{1+z^2} = \frac{\lim_{z\to 0} z - z^2}{\lim_{z\to 0} 1 + z^2} = \frac{0}{1} = 0$$