Homework 2 MATH 1199 Fall 2019

(1a) $z_1 = 2+2i$, $z_2 = 3i$

Soln: $z_1 = |2+2i| e^{i \arctan(\frac{2}{2})} = \sqrt{8}\, e^{i\frac{\pi}{4}}$

$z_2 = |3i| e^{i\frac{\pi}{2}} = 3e^{i\frac{\pi}{2}}$

So compute

$$z_1 z_2 = 3\sqrt{8}\, e^{i(\frac{\pi}{4}+\frac{\pi}{2})} = 3\sqrt{8}\, e^{i\frac{3\pi}{4}}$$

$$= 3\sqrt{8}\left[\cos\left(\tfrac{3\pi}{4}\right) + i\sin\left(\tfrac{3\pi}{4}\right)\right]$$

$\sqrt{8} = 2\sqrt{2}$

$$= 3\sqrt{8}\left[-\tfrac{\sqrt{2}}{2}\right] + 3\sqrt{8}\left[\tfrac{\sqrt{2}}{2}\right] i$$

$$= -6+6i$$

(1b) $z_1 = -\sqrt{3} - i = |-\sqrt{3}-i| e^{i(\pi + \arctan(\frac{-1}{-\sqrt{3}}))} = 2e^{\frac{7\pi}{6}i}$

$z_2 = 5 = |5| e^{0i} = 5$

So,

$$z_1 z_2 = 10e^{\frac{7\pi}{6}i} = 10\cos\left(\tfrac{7\pi}{6}\right) + 10\sin\left(\tfrac{7\pi}{6}\right)i$$

$$= 10\left(-\tfrac{\sqrt{3}}{2}\right) + 10\left(\tfrac{-1}{2}\right)i$$

$$= -5\sqrt{3} - 5i$$

(1c) $z_1 = 1-i = |1-i| e^{i\arctan(-1)} = \sqrt{2}\, e^{-\frac{\pi}{4}i}$

$z_2 = 5+5i = |5+5i| e^{i\arctan(1)} = \sqrt{50}\, e^{i\frac{\pi}{4}}$

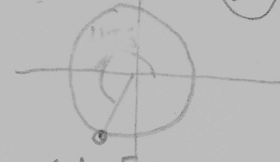
So,

$$z_1 z_2 = \sqrt{100}\, e^{0i} = 10$$

(1d) $z_1 = 3i = 3e^{\frac{\pi}{2}i}$ $\Rightarrow$ $z_1 z_2 = 12e^{0i} = 12$

$z_2 = -4i = 4e^{-\pi/2\, i}$

(2) $\frac{2i}{1+i} = \frac{2i}{1+i}\left(\frac{1-i}{1-i}\right) = \frac{2i-2i^2}{1-i^2} = \frac{2+2i}{2} = 1+i$



$\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

Therefore,

$$\operatorname{Arg}\left(\frac{2i}{1+i}\right) = \operatorname{Arg}(1+i) = \arctan(1) = \frac{\pi}{4}$$

(3a) $(-1+\sqrt{3}i)^{14} = \left(2e^{\frac{2\pi}{3}i}\right)^{14} = 2^{14} e^{\frac{28\pi}{3}i}$

$= 2^{14}\left[\cos\left(\frac{28\pi}{3}\right) + i\sin\left(\frac{28\pi}{3}\right)\right]$

$= 2^{14}\left(-\frac{1}{2}\right) + 2^{14}\left(-\frac{\sqrt{3}}{2}\right)i$

$= -2^{13} - 2^{13}\sqrt{3}\,i$

$2\pi = \frac{6\pi}{3}$

$4\pi = \frac{12\pi}{3}$

$6\pi = \frac{18\pi}{3}$

$8\pi = \frac{24\pi}{3}$

$\Downarrow$

$\frac{28\pi}{3} = 8\pi + \frac{4\pi}{3}$

(3b) $(-2-2\sqrt{3}i)^{11} = \left(4e^{\frac{4\pi}{3}i}\right)^{11} = 4^{11} e^{\frac{44\pi}{3}i}$

$= 4^{11}\left[\cos\left(\frac{44\pi}{3}\right) + \sin\left(\frac{44\pi}{3}\right)i\right]$

$= 4^{11}\left[\cos\left(14\pi + \frac{2\pi}{3}\right) + i\sin\left(14\pi + \frac{2\pi}{3}\right)\right]$

$= 4^{11}\left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right]$

b/c cos + sin are 2π-periodic

$= \frac{-4^{11}}{2} + \frac{4^{11}\cdot\sqrt{3}}{2}i$

$|-2-2\sqrt{3}| = \sqrt{4+12} = \sqrt{16} = 4$

$\pi + \arctan\left(\frac{-2\sqrt{3}}{-2}\right)$

$= \pi + \arctan(\sqrt{3})$

$= \pi + \frac{\pi}{3}$

$= \frac{4\pi}{3}$

$2\pi = \frac{6\pi}{3}$

$4\pi = \frac{12\pi}{3}$

$6\pi = \frac{18\pi}{3}$

$8\pi = \frac{24\pi}{3}$

$10\pi = \frac{30\pi}{3}$

$12\pi = \frac{36\pi}{3}$

$14\pi = \frac{42\pi}{3}$

$\Downarrow$

44π

(4) $1^{1/5} = \left(e^{0\pi i}\right)^{1/5} = e^{\left(\frac{0}{5} + \frac{2k\pi}{5}\right)i}$

5^{th} roots $\rightsquigarrow \left\{1, e^{\frac{2\pi}{5}i}, e^{\frac{4\pi}{5}i}, e^{\frac{6\pi}{5}i}, e^{\frac{8\pi i}{5}}\right\}$

(5) $1^{1/6} = \left(e^{0\pi i}\right)^{1/6} = e^{\left(\frac{0}{6} + \frac{2k\pi}{6}\right)i}$

6^{th} roots $\rightsquigarrow \left\{1, e^{\frac{2\pi}{6}i}, e^{\frac{4\pi}{6}i}, \underset{-1}{e^{\frac{6\pi}{6}i}},\right.$

6) $i^{1/3} = \left(e^{\frac{\pi}{2}i}\right)^{1/3} = e^{\left(\frac{\pi}{6} + \frac{2k\pi i}{3}\right)} = e^{\left(\frac{\pi}{6} + \frac{4k\pi i}{6}\right)}$

$a+bi$

So the cube roots of i are

$k=0$: $e^{\frac{\pi}{6}i} = \cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)$

$= \frac{\sqrt{3}}{2} + \frac{1}{2}i$

$k=1$: $e^{\frac{5\pi}{6}i} = \cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2} + \frac{i}{2}$

$k=2$: $e^{\frac{9\pi i}{6}} = \cos\left(\frac{9\pi i}{6}\right) + i\sin\left(\frac{9\pi i}{6}\right) = 0 - i$

7) $(1+i)^{1/4} = \left(\sqrt{2}e^{\frac{\pi}{4}i}\right)^{1/4} = 2^{\frac{1}{8}} e^{\frac{\pi}{16}i + \frac{2\pi i k}{4}} = 2^{\frac{1}{8}} e^{\frac{\pi}{16}i + \frac{4\pi i k}{16}}$

$2^{1/2}$

So 4th roots are $\left\{2^{\frac{1}{8}}e^{\frac{\pi}{16}i},\ 2^{\frac{1}{8}}e^{\frac{5\pi i}{16}},\ 2^{\frac{1}{8}}e^{\frac{9\pi i}{16}},\ 2^{\frac{1}{8}}e^{\frac{13\pi i}{16}}\right\}$

8) $(-\sqrt{3}-i)^{1/3} = \left(2e^{-\frac{5\pi}{6}i}\right)^{1/3} = 2^{1/3} e^{\left(-\frac{5\pi}{18} + \frac{2\pi k}{3}\right)i} = 2^{1/3} e^{\left(-\frac{5\pi}{18} + \frac{12\pi k}{18}\right)i}$

roots: $\left\{2^{1/3}e^{-\frac{5\pi}{18}i},\ 2^{1/3}e^{\frac{7\pi i}{18}},\ 2^{1/3}e^{\frac{19\pi i}{18}}\right\}$

9) $(-1)^{1/4} = \left(e^{\pi i}\right)^{1/4} = e^{\left(\frac{\pi}{4} + \frac{2k\pi}{4}\right)i}$

roots: $\left\{e^{\frac{\pi}{4}i},\ e^{\frac{3\pi}{4}i},\ e^{\frac{5\pi}{4}i},\ e^{\frac{7\pi i}{4}}\right\}$

10a

$$f(x+iy) = (x+iy)^2 - 2(x-iy)$$
$$= (x^2-y^2) + 2xyi - 2x + 2yi$$
$$= [x^2-y^2-2x] + [2xy-2y]i$$

10b

$$g(x+iy) = \frac{x+iy}{x+iy+1} = \frac{(x+iy)}{(x+iy+1)}\overbrace{\left(\frac{x+1-iy}{x+1-iy}\right)}^{=1}$$
$$= \frac{x^2+x-\cancel{ixy}+\cancel{xyi}+iy-i^2y^2}{(x+1)^2-i^2y^2}$$
$$= \left(\frac{x^2+y^2+x}{(x+1)^2+y^2}\right) + \left(\frac{y^2}{(x+1)^2+y^2}\right)i$$

(10c)

$$h(x+iy) = \frac{(x+iy)^2}{x+iy-1-i} = \frac{(x^2-y^2)+xyi}{(x-1)+(y-1)i}\left(\frac{(x+1)-(y-1)i}{(x+1)-(y-1)i}\right)$$
$$= \frac{(x^2-y^2)(x+1) - (x^2-y^2)(y-1)i + xy(x+1)i - xy(y-1)i^2}{(x-1)^2-(y-1)^2y}$$
$$= \left[\frac{(x^2-y^2)(x+1)+xy(y-1)}{(x-1)^2-(y-1)^2y}\right] + \left[\frac{-(x^2-y^2)(y-1)+xy(x+1)}{(x-1)^2-(y-1)^2y}\right]i$$

(10d)

$$l(x+iy) = \frac{\overline{x+iy}}{|x+iy|} = \frac{x-iy}{\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}} - \frac{y}{\sqrt{x^2+y^2}}i$$