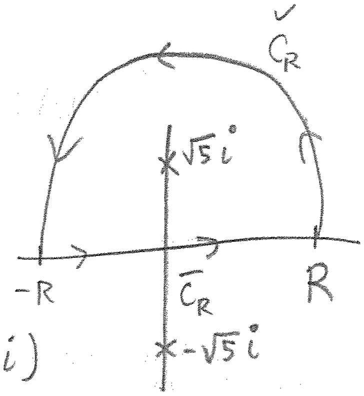
HW11 MATH 1199 Fall 2019

(1) $\displaystyle\int_{-\infty}^{\infty} \frac{3}{x^2+5}\,dx$



$C_R = \check{C}_R \cup \bar{C}_R$

Soln: Poles: $x^2+5=0 \to x = \pm\sqrt{5}\,i$

$\to x^2+5 = (x-\sqrt{5}\,i)(x+\sqrt{5}\,i)$

$$(\star)_R \quad \int_{C_R} \frac{3}{z^2+5}\,dz = \int_{\check{C}_R} \frac{3}{z^2+5}\,dz + \underbrace{\int_{\bar{C}_R} \frac{3}{z^2+5}\,dz}_{=\int_{-R}^{R} \frac{3}{z^2+5}\,dz}$$

$$\int_{C_R} \frac{3}{z^2+5}\,dz \overset{\substack{\text{Residue}\\ \text{thm}}}{=} 2\pi i \operatorname*{Res}_{z=\sqrt{5}\,i} \frac{\frac{3}{z+\sqrt{5}\,i}}{z-\sqrt{5}\,i} = \cancel{2}\pi \cancel{i}\left(\frac{3}{\cancel{2}\sqrt{5}\,\cancel{i}}\right) = \frac{3\pi}{\sqrt{5}}$$

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for $z \in \check{C}_R$, $|z| = R$ so for such z,

$$|z^2+5| \geq \big||z|^2 - |5|\big| = |R^2-5|, \text{ hence } \left|\frac{3}{z^2+5}\right| \leq \frac{3}{|R^2-5|} = M$$

also $\check{C}_R$ has length $\pi R = L$. Therefore by ML-inequality,

$$\left|\int_{\check{C}_R} \frac{3}{z^2+5}\,dz\right| \leq ML = \frac{3\pi R}{|R^2-5|} \xrightarrow{R\to\infty} 0, \text{ hence}$$

$\left|\int_{\check{C}_R} \frac{3}{z^2+5}\,dz\right| \to 0$ as $R\to\infty$, therefore, taking limit as $R\to\infty$ of $(\star)_R$,

we get

$$\boxed{\frac{3\pi}{\sqrt{5}} = \int_{-\infty}^{\infty} \frac{3}{z^2+5}\,dz}$$