Homework 10 MATH 1199 Fall 2019

1

① $f(z) = \dfrac{1+5z^3}{z^4+z^7} = \dfrac{1}{z^4}\left(\dfrac{1+5z^3}{1+z^3}\right) = \dfrac{1}{z^4}\left(\dfrac{5(1+z^3)-4}{1+z3}\right) = \dfrac{1}{z^4}\left(5 - \dfrac{4}{1+z^3}\right)$

So, from

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k, \quad |z|<1$$

$\downarrow$ replace $z \mapsto -z$ $\downarrow$

$$\frac{1}{1+z} = \sum_{k=0}^{\infty} (-1)^k z^k, \quad |-z| = |z| < 1$$

$\downarrow$ replace $z \mapsto z^3$ $\downarrow$ $(z^3)^k = z^{3k}$

$$\frac{1}{1+z^3} = \sum_{k=0}^{\infty} (-1)^k z^{3k}, \quad |z^3|<1 \Leftrightarrow |z|<1$$

Therefore, we have

$$f(z) = \frac{1}{z^4}\left(5 - \frac{4}{1+z^3}\right)$$

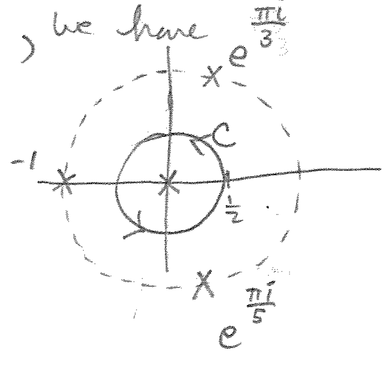
$$= \frac{1}{z^4}\left(5 - 4\sum_{k=0}^{\infty}(-1)^k z^{3k}\right), \quad |z|<1$$

$$= \frac{5}{z^4} - 4\sum_{k=0}^{\infty}(-1)^k z^{3k-4}, \quad |z|<1$$

$$= \frac{5}{z^4} - \frac{4}{z^4} + \frac{4}{z} - 4\sum_{k=2}^{\infty}(-1)^k z^{3k-4}$$

$$= \frac{1}{z^4} + \frac{4}{z} - 4\sum_{k=2}^{\infty}(-1)^k z^{3k-4}$$

If C is $|z| = \frac{1}{2}$, we have



$1+z^3 = 0$

$z^3 = -1$

$z = e^{\frac{i\pi + 2\pi i k}{3}}, k=0,1,2$

$= e^{\frac{\pi i}{3}}, e^{\pi i} = -1, e^{\frac{5\pi i}{3}}$

Since 0 only blow-up point inside C, and coeff of $\frac{1}{z}$ in series expansion is 4,

$$4 \underset{\text{series}}{=} c_{-1} \underset{\text{Thm}}{=} \frac{1}{2\pi i}\int_C f(z)\,dz$$

$\Downarrow$

$$\int_C \frac{1+5z^3}{z^4+z^7}\,dz = 8\pi i$$

(2) $e^{z} = \sum_{k=0}^{\infty} \frac{z^k}{k!}, \quad z \in \mathbb{C}$

$\downarrow$ replace $z \mapsto \frac{1}{z^4}$ $\downarrow$

$e^{\frac{1}{z^4}} = \sum_{k=0}^{\infty} \frac{1}{z^{4k}\, k!}, \quad \frac{1}{z} \in \mathbb{C} \leftrightarrow z \in \mathbb{C}\setminus\{0\}$

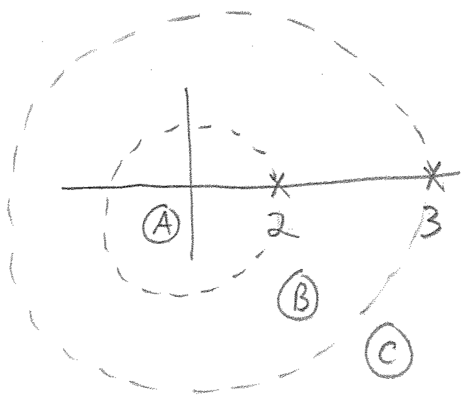
$= 1 + \frac{1}{z^4} + \frac{1}{2z^8} + \cdots$

In this case, coefficient of $\frac{1}{z}$ is zero (it doesn't appear), so

$$0 = c_{-1} = \frac{1}{2\pi i} \int_C e^{\frac{1}{z^4}}\, dz$$

$$\Rightarrow \int_C e^{\frac{1}{z^4}}\, dz = 0$$

(3) $f(z) = \frac{1}{(z-2)(z-3)}$ $\rightsquigarrow$



Let (A) denote $|z| < 2$,

let (B) denote $2 < |z| < 3$, and

let (C) denote $|z| > 3$.

First use partial fractions:

$$\frac{1}{(z-2)(z-3)} = \frac{A}{z-2} + \frac{B}{z-3}$$

$1 = A(z-3) + B(z-2)$

$1 = (A+B)z + (-3A-2B) \Longrightarrow \begin{cases} A+B=0 \longrightarrow A=-B \\ -3A-2B=1 \longrightarrow 3B-2B=1 \end{cases}$

$B = 1 \longrightarrow A = -1$

$$\Rightarrow \frac{1}{(z-2)(z-3)} = \frac{-1}{z-2} + \frac{1}{z-3}$$

For Ⓐ ~ $|z|<2$

(i) $\frac{-1}{z-2} = \frac{1}{2-z} = \frac{1/2}{1-(z/2)} \overset{\text{geo series}}{\Longrightarrow} \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{z}{2}\right)^k = \sum_{k=0}^{\infty}\frac{z^k}{2^{k+1}}$, valid for $\left|\frac{z}{2}\right|<1$, i.e. $|z|<2$

(ii) $\frac{1}{z-3} = \frac{-1}{3-z} = \frac{-1/3}{1-(z/3)} \overset{\text{geo series}}{\Longrightarrow} -\frac{1}{3}\sum_{k=0}^{\infty}\left(\frac{z}{3}\right)^k = -\sum_{k=0}^{\infty}\frac{z^k}{3^{k+1}}$; valid for $\left|\frac{z}{3}\right|<1$, i.e. $|z|<3$

Therefore (i) and (ii) hold in Ⓐ and we can write

$$f(z) = \frac{-1}{z-2} + \frac{1}{z-3} = \sum_{k=0}^{\infty}\left(\frac{1}{2^{k+1}} - \frac{1}{3^{k+1}}\right)z^k$$, valid for intersection of $|z|<2$ and $|z|<3$, i.e. valid for $|z|<2$

For Ⓑ ~ $2<|z|<3$

The series (ii) we found is still valid here, but (i) is not. So,

(iii) $\frac{-1}{z-2} = \frac{-1/z}{1-\frac{2}{z}} = -\frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{2}{z}\right)^k = -\sum_{k=0}^{\infty}\frac{2^k}{z^{k+1}} \overset{\text{reindex}}{=} -\sum_{k=1}^{\infty}\frac{2^{k-1}}{z^k}$, valid for $\left|\frac{2}{z}\right|<1$, i.e. $\left|\frac{z}{2}\right|>1$, i.e. $|z|>2$

Therefore (ii) and (iii) hold in Ⓑ and we can write

$$f(z) = -\frac{1}{z-2} + \frac{1}{z-3} = -\sum_{k=1}^{\infty}\frac{2^{k-1}}{z^k} - \sum_{k=0}^{\infty}\frac{z^k}{3^{k+1}}$$

For Ⓒ

The series (iii) is still valid but (ii) is not. So,

(iv) $\frac{1}{z-3} = \frac{1/z}{1-(3/z)} = \frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{3}{z}\right)^k = \sum_{k=0}^{\infty}\frac{3^k}{z^{k+1}} \overset{\text{reindex}}{=} \sum_{k=1}^{\infty}\frac{3^{k-1}}{z^k}$, valid for $\left|\frac{3}{z}\right|<1$, i.e. $\left|\frac{z}{3}\right|>1$, i.e. $|z|>3$

Therefore (iii) and (iv) hold in Ⓑ and we can write

$$f(z) = \frac{-1}{z-2} + \frac{1}{z-3} = -\sum_{k=1}^{\infty}\frac{2^{k-1}}{z^k} + \sum_{k=1}^{\infty}\frac{3^{k-1}}{z^k} = \sum_{k=1}^{\infty}\frac{\left(3^{k-1}-2^{k-1}\right)}{z^k}$$, valid for $|z|>3$.

(4)

(a) $$\frac{1}{z(1+z)} = \frac{1}{z} \underbrace{\sum_{k=0}^{\infty} (-1)^k z^k}_{= \frac{1}{1+z}}$$ ~ we know now from earlier (see Problem 1)

$$= \sum_{k=0}^{\infty} (-1)^k z^{k-1} = \frac{1}{z} - 1 + z^2 - z^3 + \ldots$$

Therefore,

$$\operatorname{Res}_{z=0} \frac{1}{z(z+1)} = \text{coeff of } \frac{1}{z} \text{ in Laurent series centered at } 0 = 1$$

(b) $$z\cos\left(\frac{1}{z}\right) = z \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!\,z^{2k}} = \sum_{k=0}^{\infty} \frac{(-1)^k z}{(2k)!\,z^{2k}} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!\,z^{2k-1}}$$

$$= \frac{1}{0!\,z^{-1}} - \frac{1}{2!\,z} + \frac{1}{4!\,z^3} - \ldots$$

$$= z - \frac{1}{2z} + \frac{1}{24z^3} - \ldots$$

Therefore,

$$\operatorname{Res}_{z=0} z\cos\left(\frac{1}{z}\right) = \text{coeff of } \frac{1}{z} \text{ in Laurent series centered at } 0 = -\frac{1}{2}$$

(c) $$\frac{z - \sin(z)}{z} = 1 - \frac{1}{z} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!}$$

$$= 1 - \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k+1)!}$$

$$= 1 - 1 + \frac{z^2}{3!} - \frac{z^4}{5!} + \ldots$$

Therefore,

$$\operatorname{Res}_{z=0} \frac{z - \sin(z)}{z} = \text{coeff of } \frac{1}{z} \text{ in Laurent series centered at } 0 = 0$$