Quiz 3 — MATH 4590 Spring 2018

1. Let (M, d_1) and (N, d_2) be metric spaces and fix some $y \in N$. Define $f\colon M \to N$ to be the constant function $f(x) = y$. Prove that f is continuous.
   Proof: Let $\epsilon > 0$, let $p \in M$, and choose $\delta > 0$ to be any positive real number. Then if $q \in M$ with $d_1(p, q) < \delta$, we may compute
   $$d_2(f(p), f(q)) = d_2(y, y) = 0 < \epsilon,$$
   completing the proof. ■

2. Let (M, d_1) be a metric space and let $(N, d_2) = (\mathbb{R}, d)$, where $d(x, y) = |x - y|$. Let $f\colon M \to \mathbb{R}$ and $g\colon M \to \mathbb{R}$ be continuous. Prove that the function $h\colon M \to \mathbb{R}$ given by $h(x) = f(x) + g(x)$ is continuous.
   Proof: Let $\epsilon > 0$ and let $p \in M$. Since f is continuous, there is a number $\delta_f > 0$ with the property that for $q \in M$ with $d_1(p, q) < \delta_f$, it follows that
   $$d_2(f(p), f(q)) = |f(p) - f(q)| < \frac{\epsilon}{2}.$$
   Similarly, there is a number $\delta_g > 0$ such that if $q \in M$ with $d_1(p, q) < \delta_g$, it follows that
   $$d_2(g(p), g(q)) = |g(p) - g(q)| < \frac{\epsilon}{2}.$$
   Choose δ so that $0 < \delta < \min\{\delta_f, \delta_g\}$. Now calculate
   $$\begin{aligned} d_2(h(p), h(q)) &= |h(p) - h(q)| \\ &= |(f(p) + g(p)) - (f(q) + g(q))| \\ &= |(f(p) - f(q)) + (g(p) - g(q))| \\ &\leq |f(p) - f(q)| + |g(p) - g(q)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon, \end{aligned}$$
   completing the proof. ■