MATH 4590 – Study for exam 3

Definitions

partition pair, mesh of a partition, Riemann sum, Riemann integral, upper sum, lower sum, upper integral, lower integral, Darboux integrable, refinement of a partition, zero set, antiderivative, indefinite integral, pointwise convergence of sequence of functions, uniform convergence of sequence of functions

Stuff from homework

(HW8): be able to compute Riemann sums and upper and lower sums on the picture of a graph (a calculator will be provided if this comes up), be able to handle proofs involving the definition of Riemann or Darboux integrability (such as those in numbers 4 and 5)

(HW9): induction proof about sums, drawing piecewise functions and using them to compute integrals, proofs that a described set is a zero set, proofs with the antiderivative theorem and fundamental theorem of calculus, simple proofs involving uniform convergence

(HW10): calculations showing the necessity of uniform convergence as an assumption on a sequence that "preserves nice properties" (such as numbers 1,3,5), finding a counterexample (like number 2)

Theorems to know: If f is Riemann integrable, then f is bounded. The integral is linear, i.e.

$$\int \alpha f(x) + \beta g(x)\mathrm{d}x = \alpha \int f(x)\mathrm{d}x + \beta \int g(x)\mathrm{d}x.$$

If $f(x) \leq g(x)$ for all x, then $\displaystyle\int f(x)\mathrm{d}x \leq \int g(x)\mathrm{d}x$. If for all x, $|f(x)| \leq M$, then $\displaystyle\left|\int_a^b f(x)\mathrm{d}x\right| \leq M(b-a)$. The "refinement principle": refining a partition causes the lower integral to increase and the upper integral to decrease. Being "Riemann integrable" and being "Darboux integrable" are equivalent. Riemann integrability criterion: A bounded function is Riemann-integrable if and only if $\forall \epsilon > 0 \exists P$ such that $U(f,P) - L(f,P) < \epsilon$.

$$\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^d f(x)\mathrm{d}x$$

The following kinds of functions are Riemann-integrable: continuous, bounded piecewise-continuous, monotone, product of Riemann-integrable functions, composition of Riemann-integrable functions, the absolute value of a Riemann-integrable function. The fundamental theorem of calculus: if $f\colon [a,b] \to \mathbb{R}$ is Riemann-integrable, then its indefinite integral $F\colon [a,b] \to \mathbb{R}$ is continuous and $F'(x) = f(x)$. Antiderivative theorem: if $f\colon [a,b] \to \mathbb{R}$ is Riemann-integrable and G is any antiderivative of f (i.e. $G'(x) = f(x)$), then there is a constant C so that

$$G(x) = \int_a^x f(t)\mathrm{d}t + C.$$

Cauchy Criterion for uniform convergence: $f_n \rightrightarrows f$ if and only if $\forall \epsilon > 0 \exists N \in \mathbb{N}$ so that if $m, n \geq N$ and $x \in [a,b]$, then $|f_n(x) - f_m(x)| < \epsilon$. If $f_n \rightrightarrows f$, $\displaystyle\lim_{n\to\infty} f_n(x) = A_n$, and A_n is a convergent sequence, then

$$\lim_{t\to x} f(t) = \lim_{n\to\infty} A_n.$$

If $f_n \rightrightarrows f$ and each f_n is Riemann-integrable, then so is f and moreover

$$\int_a^b f(x)\mathrm{d}x = \lim_{n\to\infty} \int_a^b f_n(x)\mathrm{d}x.$$

Suppose f_n is a sequence of differentiable functions and there exists some x_0 so that $f_n(x_0)$ converges as $n \to \infty$. If f_n' converges uniformly to a function, then f_n converges uniformly to a function f, and moreover,

$$f'(x) = \lim_{n\to\infty} f_n'(x).$$

Stuff from quizzes

Quiz 7: proving that uniform convergence implies pointwise convergence

Quiz 8: calculations that justify the definition of uniform convergence (from slides)
Quiz 9: calculations that justify the definition of uniform convergence (from slides)
Quiz 10: calculations that justify the definition of uniform convergence (from slides)