continuity: Let (M, d_1) and (N, d_2) be metric spaces. We say that a function $f\colon M \to N$ is continuous provided that

$$\forall \epsilon > 0 \forall p \in M \exists \delta > 0 \text{ such that } q \in M \text{ and } d_1(p,q) \implies d_2(f(p), f(q)) < \epsilon$$

Example: Consider the metric space $\mathbb{R}$ endowed with the usual metric $d(x,y) = |x-y|$. Show that $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = 8x^2 - 3x + 2$ is continuous.

Scratch work: Given a fixed p, we want to find a bound on δ such that a q obeying $d(p,q) = |p-q| < \delta$ implies

$$(*) \qquad d(f(p), f(q)) = |f(p) - f(q)| = |(8p^2 - 3p + 2) - (8q^2 - 3q + 2)| < \epsilon.$$

To find the condition on δ, we do algebra. Our algebra has a goal: we only have "control" over $|p-q|$ (δ cannot depend on q – in actuality, δ determines which q's are allowed!!! However, δ *often does* depend on p).

Compute:

$$\begin{aligned}
|(8p^2 - 3p + 2) - (8q^2 - 3q + 2)| &= |8(p^2 - q^2) - 3(p-q)| \\
&= |8\underbrace{(p-q)(p+q)}_{\text{factor}} - 3(p-q)| \\
&= \underbrace{|p-q|}_{\text{we control}} \Big| \underbrace{8(p+q) - 3}_{\text{want } q-p \text{ (why not } p-q?)} \Big| \\
&= |p-q||8(q + p \underbrace{-p + p}_{\text{add zero}}) - 3| \\
&= |p-q| \underbrace{|8(q-p) + 16p - 3|}_{\text{prepare for triangle inequality}} \\
&\underbrace{\leq}_{\text{triangle inequality}} |p-q|\Big[8|q-p| + |16p-3|\Big] \\
&\underbrace{<}_{\text{we control } |p-q|<\delta} \delta\Big[8\delta + |16p-3|\Big] \\
&\underbrace{<}_{\text{we can choose } \delta<1} \underbrace{\delta}_{\text{leave this one}} \Big[8 + |16p-3|\Big]
\end{aligned}$$

Our ultimate goal is to find a condition on δ that causes $(*)$ to occur. If we pick the number δ so that

$$|(8p^2 - 3p + 2) - (8q^2 - 3q + 2)| < \delta\Big[8 + |16p-3|\Big| < \epsilon,$$

then it will work. In other words, "solve" for δ in the inequality

$$\delta\Big[8 + |16p-3|\Big] < \epsilon.$$

Thus, take

$$\delta < \frac{\epsilon}{8 + |16p-3|}.$$

Proof: Let $\epsilon > 0$ and let $p \in \mathbb{R}$. Choose $0 < \delta < \dfrac{\epsilon}{8 + |16p-3|}$. Then if $q \in \mathbb{R}$ with $d(q,p) = |q-p| < \delta$, we compute

$$|(8p^2 - 3p + 2) - (8q^2 - 3q + 2)| \quad \leq \delta\Big[8 + |16p-3|\Big] < \left(\frac{\epsilon}{8 + |16p-3|}\right)(8 + |16p-3|) = \epsilon,$$

completing the proof.