Pointwise and uniform convergence

Definition ("pointwise convergence"): Let $f_n\colon [a,b] \to \mathbb{R}$ be a sequence of functions. We say that f_n converges pointwise to f provided that $\forall x \in [a,b] \forall \epsilon > 0 \exists N \in \mathbb{N}$ so that for all $n \geq N, |f_n(x) - f(x)| < \epsilon$. For this, we write $f_n \to f$.

Definition ("uniform convergence"): Let $f_n\colon [a,b] \to \mathbb{R}$ be a sequence of functions. We say that f_n converges uniformly to f provided that $\forall \epsilon > 0 \exists N \in \mathbb{N}$ so that for all $n \geq N$ and $x \in [a,b]$, $|f_n(x) - f(x)| < \epsilon$. For this, we write $f_n \rightrightarrows f$.

Four questions

Let $f_n\colon [a,b] \to \mathbb{R}$ be a sequence of functions. Let define the function

$$F(x) = \sum_{k=0}^{\infty} f_n(x),$$

and assume that the series always converges (so this definition makes sense).

1. What conditions guarantee that F is continuous?
2. What conditions guarantee that F is differentiable?
3. What conditions allow us to write

$$\int_a^b F(x)\mathrm{d}x = \int_a^b f_1(x)\mathrm{d}x + \int_a^b f_2(x)\mathrm{d}x + \ldots ?$$

4. What conditions allow us to write $F'(x) = f_1'(x) + f_2'(x) + \ldots$?

From last class

Last Friday: we argued that the sequence $f_n \colon [0,1] \to \mathbb{R}$ defined by $f_n(x) = x^n$ was pointwise convergent to zero (i.e. $f_n \to 0$) but not uniformly convergent to zero, (i.e. $\not\rightrightarrows 0$).

Four similar questions

Let $\mathcal{F}(x) = \lim_{n\to\infty} f_n(x)$.

1.' What conditions on f_n will guarantee that $\mathcal{F}$ is cotinuous?

2.' What conditions on f_n will guarantee that $\mathcal{F}$ is differentiable?

3.' What conditions on f_n will guarantee that

$$\int_a^b \mathcal{F}(x)\mathrm{d}x = \lim_{n\to\infty} \int_a^b f_n(x)\mathrm{d}x?$$

4.' What conditions on f_n will guarantee that $\mathcal{F}'(x) = \lim_{n\to\infty} f_n'(x)$?

In-class problem

Let $f_n \colon [0,1] \to \mathbb{R}$ be defined by $f_n(x) = \dfrac{x^n}{n}$ for $n = 1, 2, 3, \ldots$.

What is $\mathcal{F}(x)$?

What is $\mathcal{F}'(x)$?
What is $f_n'(x)$?

What is $\lim_{n\to\infty} f_n(x)$?

In-class problem

Let $f_n\colon [0,1] \to \mathbb{R}$ be defined by $f_n(x) = \dfrac{x^n}{n}$ for $n = 1, 2, 3, \ldots$.

What is $\mathcal{F}(x)$? **Answer**: $\mathcal{F}(x) = \lim_{n\to\infty} \dfrac{x^n}{n} = 0$.

What is $\mathcal{F}'(x)$? **Answer**: $\mathcal{F}'(x) = 0$.

What is $f_n'(x)$? **Answer**: $f_n'(x) = x^{n-1}$.

What is $\lim_{n\to\infty} f_n(x)$? **Answer**: $\lim_{n\to\infty} f_n'(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1. \end{cases}$

Hence:

$$\mathcal{F}'(x) \neq \lim_{n\to\infty} f_n'(x).$$

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Examples

Let $f_n \colon [0,1] \to \mathbb{R}$ be defined by

$$f_n(x) = \begin{cases} 4n^2x, & 0 \leq x \leq \dfrac{1}{2n} \\ -4n^2x + 4n, & \dfrac{1}{2n} \leq x \leq \dfrac{1}{n} \\ 0, & \dfrac{1}{n} \leq x \leq 1. \end{cases}$$

Draw f_n for some n and compute $\displaystyle\int_0^1 f_n(x)$ for your value of n.
What is $\displaystyle\lim_{n\to\infty} f_n(x)$ for any value of x?

References

"Advanced Calculus" by A.E. Taylor