Series

Recall:

- $\lim_{n\to\infty} p_n = p$ means $\forall \epsilon > 0 \exists N \in \mathbb{N}$ so that if $n \geq N$, then $|p_n - p| < \epsilon$.
- Cauchy sequence a sequence (p_n) is called a Cauchy sequence if $\forall \epsilon > 0 \exists N \in \mathbb{N}$ so that $n, m > N$ implies $|p_n - p_m| < \epsilon$.
- We saw earlier: "a sequence of real numbers is convergent if and only if it is a Cauchy sequence"

We say that $\sum_{k=0}^{\infty} A_k$ converges to A provided that

$$\lim_{n\to\infty} \sum_{k=0}^{n} A_k.$$

exists and equals A.

Series

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Series - Cauchy condition

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Cauchy Criterion for Convergence ("CCC"): The series $\sum_{k=0}^{\infty} a_k$ converges if and only if $\forall \epsilon > 0 \exists N \in \mathbb{N}$ such that if $n \geq m \geq N$, then $\left| \sum_{k=m}^{n} a_k \right| < \epsilon$

Comparison test

Theorem: If $\sum b_k$ "dominates" $\sum a_k$ in the sense that for all sufficiently large k, $|a_k| \leq b_k$, then whenever $\sum b_k$ converges, it follows that $\sum a_k$ converges.

Proof: Since $\sum b_k$ converges, there is an $N \in \mathbb{N}$ so that if $n \geq m \geq N$, then $\left|\sum_{k=m}^{n} b_k\right| < \epsilon$. So, calculate

$$\left|\sum_{k=m}^{n} a_k\right| \leq \sum_{k=m}^{n} |a_k| \leq \sum_{k=m}^{n} b_k < \epsilon.$$

Therefore by **CCC**, $\sum a_k$ converges. ■

Integral test

Theorem: Suppose that $\int_0^\infty f(x)\mathrm{d}x$ is an improper integral and $\sum a_k$ is a given series. Then

a) If $|a_k| \leq f(x)$ for all sufficiently large k and all $x \in (k-1, k]$, then the convergence of the improper integral implies convergence of the series.
b) If $|f(x)| \leq a_k$ for all sufficiently large k and all $x \in [k, k+1)$, then divergence of the improper integral implies divergence of the series.

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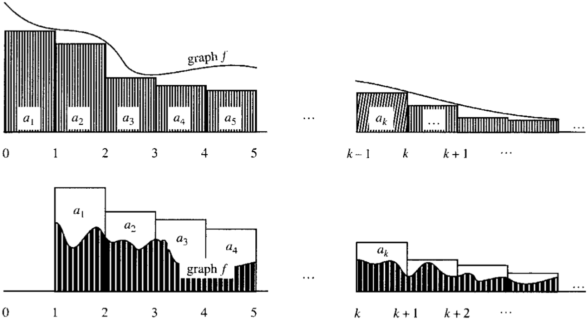
Proof:



Other tests

Corollary: The p-series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

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Theorem (Root test): Consider the series $\sum a_k$. Let $\alpha = \lim_{k \to \infty} \sqrt[k]{|a_k|}$. If $\alpha < 1$, then the series converges. If $\alpha > 1$, then the series diverges. If $\alpha = 1$, the test is inconclusive.

Theorem (Ratio test): Consider $\sum a_k$. Let $\alpha = \lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right|$. If $\alpha < 1$, then the series converges. If $\alpha > 1$, then the series diverges. If $\alpha = 1$, then the test is inconclusive.

Function spaces

Weird idea: think of a collection of functions obeying a certain property as "points" in a metric space. In this way, we can imagine what it means for a sequence of **functions** to converge to a function.

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The easiest idea of function convergence is the following.

Definition ("pointwise convergence"): Let $f_n \colon [a, b] \to \mathbb{R}$ be a sequence of functions. We say that f_n converges pointwise to f provided that $\forall x \in [a, b]$, $\lim_{n\to\infty} f_n(x) = f(x)$. In other words: $\forall x \in [a, b] \forall \epsilon > 0 \exists N \in \mathbb{N}$ so that for all $n \geq N$, $|f_n(x) - f(x)| < \epsilon$. In this situation we write $f_n \to f$ or $\lim_{n\to\infty} f_n = f$.

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Definition ("uniform convergence"): Let $f_n \colon [a, b] \to \mathbb{R}$ be a sequence of functions. We say that f_n converges uniformly to f provided that $\forall \epsilon > 0 \exists N \in \mathbb{N}$ so that for all $n \geq N$ and $x \in [a, b]$, $|f_n(x) - f(x)| < \epsilon$. In this siuation, we write $f_n \rightrightarrows f$ or $\operatorname{unif}\lim_{n\to\infty} f_n = f$.

What's the difference?

Intuition: Draw an "ϵ-tube" around the graph of f. Uniform convergence means that, for sufficiently large n, the graph of f_n lies entirely inside the ϵ-tube.

Example: Define $f_n \colon (0,1) \to \mathbb{R}$ by $f_n(x) = x^n$. For each $x \in (0,1)$,
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This example shows that there are functions that are pointwise convergence but not uniformly convergent. Are there functions that are uniformly convergent but not pointwise convergent?