Riemann integral

Definition: Let $f\colon [a, b] \to \mathbb{R}$. Let P, T be a partition pair. The Riemann sum $R(f, P, T)$ is given by

$$R(f, P, T) = \sum_{k=0}^{n} f(t_i)\Delta x_i.$$

Definition: We say f is Riemann-integrable with integral I if $\forall \epsilon > 0 \exists \delta > 0$ such that if P and T are any partition pair and $\mathrm{mesh} P < \delta$, then $|R(f, P, T) - I| < \epsilon$.

Riemann integral theorems

Theorem: If f is Riemann integrable, then f is bounded.

Theorem: $\displaystyle\int_a^b \alpha f(x) + \beta g(x) \mathrm{d}x = \alpha \int_a^b f(x) \mathrm{d}x + \beta \int_a^b g(x) \mathrm{d}x$

Theorem: If for all x, $f(x) \leq g(x)$, then $\displaystyle\int_a^b f(x) \mathrm{d}x \leq \int_a^b f(x) \mathrm{d}x$

Corollary: If for all x, $|f(x)| \leq M$, then $\displaystyle\left| \int_a^b f(x) \mathrm{d}x \right| \leq M(b - a)$.

Darboux integral

Definition: Let $f\colon [a,b] \to [-M,M]$ and let $P = \{x_1 = a < x_2 < \ldots < x_n = b\}$ be a partition of $[a,b]$. The lower sum of f is

$$L(f,P) = \sum_{i=1}^{n} m_i \Delta x_i,$$

where $m_i := \inf\{f(t)\colon x_{i-1} \le t \le x_i\}$.

Definition: The upper sum of f is

$$U(f,P) = \sum_{i=1}^{n} M_i \Delta x_i,$$

where $M_i := \sup\{f(t)\colon x_{i-1} \le t \le x_i\}$.

"Clearly":

$$L(f,P) \le R(f,P,T) \le U(f,P).$$

Darboux integral

Definition: The lower integral of f over $[a, b]$ is $\underline{I} := \sup_P L(f, P)$ and the upper integral of f over $[a, b]$ is $\underline{U} := \inf_P U(f, P)$.
Criterion: If $\underline{I} = \overline{I}$, then we say that f is Darboux-integrable over $[a, b]$ and we call this common value I.
Definition: A partition P' of $[a, b]$ is a refinement of another partition P of $[a, b]$ if $P \subset P'$.
Refinement principle: Refining a partition causes the lower sum to increase and the upper sum to decrease.
Definition: The common refinement of two partitions P and P' is the partition $P^* = P \cup P'$.
Corollary to refinement principle:

$$L(f, P) \leq L(f, P^*) \leq U(f, P^*) \leq U(f, P').$$

Consequence: A bounded function $f : [a, b] \to \mathbb{R}$ is Darboux integrable iff $\forall \epsilon > 0 \exists P$ such that $U(f, P) - L(f, P) < \epsilon$.

Riemann integral vs Darboux integral

Theorem: Riemann integrability and Darboux integrability are equivalent, i.e. functions are Riemann integrable iff they are Darboux integrable and the values of these integrals are the same.

Proof: $(\leftarrow)$ Assume f is Darboux integrable, i.e. $\underline{I} = I = \bar{I}$. Let $\epsilon > 0$. Our goal: find a $\delta > 0$ so that if P, T is a partition pair with $\mathrm{mesh}(P) < \delta$, then $|R(f, P, T) - I| < \epsilon$. Since f Darboux integrable, there is a partition P_1 of $[a, b]$ so that $U(f, P_1) - L(f, P_1) < \frac{\epsilon}{2}$. Let $\delta = \frac{\epsilon}{8Mn_1}$, where n_1 is the number of points in P_1. Let P be any partition with $\mathrm{mesh} P < \delta$. Let $P^* = P \cup P_1$ be the their common refinement. By refinement principle,

$$L(f, P_1) \leq L(f, P^*) \leq U(f, P^*) \leq U(f, P_1).$$

Subtract $L(f, P_1)$ to get

$$0 \leq L(f, P^*) - L(f, P_1) \leq U(f, P^*) - L(f, P_1) \leq \underbrace{U(f, P_1) - L(f, P_1)}_{\text{know } < \frac{\epsilon}{2}}.$$

Riemann integral vs Darboux integral

We know that $L(f, P_1) \le L(f, P^*)$, so $-L(f, P_1) \ge -L(f, P^*)$, hence

$$U(f, P^*) - L(f, P^*) \le U(f, P^*) - L(f, P_1) \underbrace{\le U(f, P_1) - L(f, P_1)}_{\text{previous slide}} < \frac{\epsilon}{2}.$$

Write $P = \{x_1, \ldots, x_n\}$ and $P^* = \{x_1^*, \ldots, x_{n^*}^*\}$. The sums $U = \sum M_i \Delta x_i$ and $U^* = \sum M_j^* \Delta x_j^*$ are identical except for terms where $x_{i-1} < x_j^* < x_i$ for some i and j. There are at most $n_1 - 2$ of these terms, and each is of magnitude $M\delta$. Thus,

$$U - U^* < (n_1 - 2)2M\delta < \frac{\epsilon}{4}.$$

Similarly, $L - L^* < \frac{\epsilon}{4}$. Thus

$$U - L = (U - U^*) + (U^* - L^*) + (L^* - L) < \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{2} = \epsilon.$$

Since $I, R(f, P, T) \in [L, U]$, we observe that $|R - I| < \epsilon$. Hence f is Riemann integrable.

Riemann integral vs Darboux integral

($\rightarrow$) Suppose that f is Riemann integrable with integral I. By Theorem, we know that f is bounded. Let $\epsilon > 0$. There exists $\delta > 0$ such that for any partition pair P, T with $\mathrm{mesh} P < \delta$, it follows that $|R(f,P,T) - I| < \frac{\epsilon}{4}$. Fix any such P. Now pick the T: choose $T = \{t_i\}$ and $T' = \{t_i'\}$ such that each $f(t_i)$ is so close to m_i and each $f(t_i')$ is so close to M_i that $|R(f,P,T) - L(f,P)| < \frac{\epsilon}{4}$ and $|U(f,P) - R(f,P,T')| < \frac{\epsilon}{4}$. Therefore

$$\begin{aligned} U(f,P) - L(f,P) &= (U(f,P) - R(f,P,T')) + (R(f,P,T') - I) \\ &\quad + (I - R(f,P,T)) + (R(f,P,T) - L(f,P)) < \epsilon. \end{aligned}$$

Since $\underline{I}, I, \overline{I} \in [L(f,P), U(f,P)]$ whose length is less than an arbitrary ϵ, the ϵ-principle shows

$$\underline{I} = I = \overline{I},$$

proving that f is Darboux-integrable. ■

Riemann Integrability Criterion

A bounded function is Riemann-integrable if and only if $\forall \epsilon > 0 \exists P$ such that $U(f, P) - L(f, P) < \epsilon$.

Example: Every continuous $f : [a, b] \to \mathbb{R}$ is integrable.
Example: A piecewise-continuous function is one which is continuous except at finitely many points. Bounded piecewise-continuous functions are Riemann-integrable.

Are there bounded functions which are not Riemann-integrable?

A bounded non-Riemann-integrable function

Let $E \subset \mathbb{R}$ and define the characteristic function (or "indicator function") $\chi_E : \mathbb{R} \to \{0, 1\}$ defined by

$$\chi_E(t) = \begin{cases} 1, & x \in E \\ 0, & x \in \mathbb{R} \setminus E. \end{cases}$$

Example: $\chi_{\mathbb{Q}}$ is not Riemann-integrable on $[0, 1]$. Why? For any partition P, $L(\chi_{\mathbb{Q}}, P) = 0$ (m_i always zero!) and $U(\chi_{\mathbb{Q}}, P) = 1$ (M_i always one!). So by the Riemann integrability criterion, the function is not Riemann-integrable!

A function with "too many" disctoninuities fails to be Riemann integrable. Can we make this more precise?

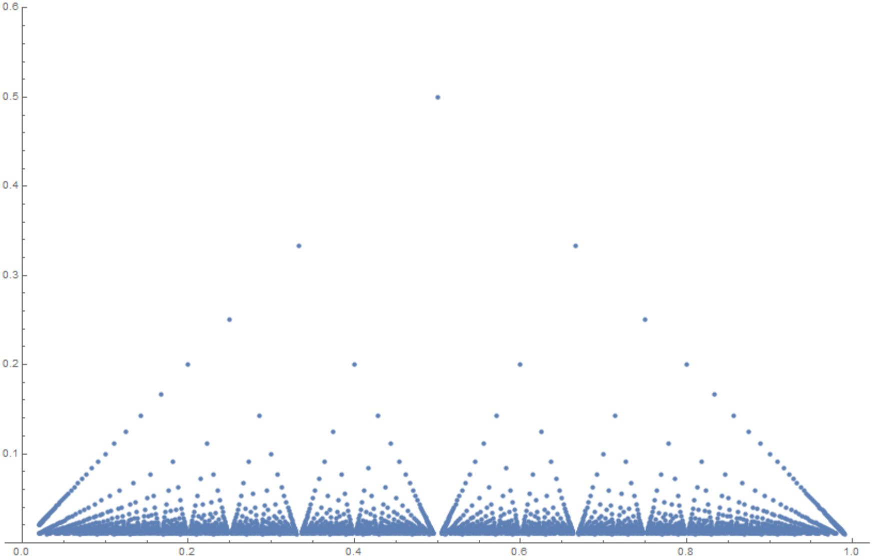
Infinitely many discontinuities yet still Riemann integrable

Fact of $\mathbb{R}$: between any two rational numbers is an irrational number and between any two irrational numbers is a rational number

Example: The Thomae function (aka "rational ruler function"...strangely enough it has found use in DNA sequencing) $f : \mathbb{R} \to \mathbb{Q}$ is defined by

$$f(x) = \begin{cases} \dfrac{1}{q}, & x = \dfrac{p}{q} \text{ is rational} \\ 0, & x \text{ is irrational.} \end{cases}$$

Infinitely many discontinuities yet still Riemann integrable



Zero sets

A set $Z \subset \mathbb{R}$ is called a zero set (or a "measure zero" set) provided that $\forall \epsilon > 0$ there are $(a_1, b_1), (a_2, b_2), \ldots$ such that $\sum_{i=1}^{\infty} b_i - a_i \leq \epsilon$. We think of zero sets as "negligible". If a property holds for all points except for those in some zero set, then we say that that property holds "almost everywhere".

Theorem: (Riemann-Lebesgue Theorem) A function $f : [a, b] \to \mathbb{R}$ is Riemann-integrable if and only if it is bounded and its set of discontinuities is a zero set.

Corollaries to Riemann-Lebesgue

Corollary: Every continuous function is Riemann-integrable.

Corollary: Every bounded piecewise-continuous function is Riemann-integrable.

Corollary: Every monotone function is Riemann-integrable.

Corollary: The product of Riemann-integrable functions is Riemann-integrable.

Corollary: The composition of Riemann-integrable functions is Riemann-integrable.

Corollary: The absolute value of a Riemann-integrable function is Riemann-integrable.

Corollary: If $a < c < b$ and $f\colon [a,b] \to \mathbb{R}$ is Riemann-integrable, then

$$\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x.$$

Corollary: If the Riemann integrable of a non-negative function f is zero, then f equals zero almost everywhere.

Fundamental theorem of calculus

Theorem: (Fundamental Theorem of Calculus) If $f\colon [a,b] \to \mathbb{R}$ is Riemann-integrable, then its indefinite integral $F\colon [a,b] \to \mathbb{R}$ defined by $F(x) = \int_a^x f(t)\mathrm{d}t$ is a continuous function of x. The derivative of $F(x)$ exists and equals $f(x)$ at all points x for which f is continuous.
Proof: Since f is Riemann-integrable, it is bounded; say for all x, $f(x) \leq |M|$. Calculate

$$|F(y) - F(x)| = \underbrace{\left|\int_a^x f(t)\mathrm{d}t - \int_a^y f(t)\mathrm{d}t\right| = \left|\int_x^y f(t)\mathrm{d}t\right|}_{\text{because } \int_a^x + \int_x^y = \int_a^y} \leq M|y - x|$$

Let $\epsilon > 0$ and choose $0 < \delta < \frac{\epsilon}{M}$. Then $|y - x| < \delta$ implies $|F(x) - F(y)| \leq M|y - x| < \epsilon$. Hence F is continuous.

Fundamental theorem of calculus

Claim: If f is continuous at x, then

$$\frac{F(x+h)-F(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\mathrm{d}t \underbrace{\longrightarrow}_{h\to 0} f(x).$$

Proof of claim: If

$$m(x,h) = \inf\{f(s)\colon |s-x| \le |h|\}$$

and

$$M(x,h) = \sup\{f(s)\colon |s-x| \le |h|\},$$

then using the fact that when x and h are fixed that $m(x,h)$ and $M(x,h)$ are constant,

$$\begin{aligned} m(x,h) &= \frac{1}{h}\int_x^{x+h} m(x,h)\mathrm{d}t \\ &\le \frac{1}{h}\int_x^{x+h} f(t)\mathrm{d}t \le \frac{1}{h}\int_x^{x+h} M(x,h)\mathrm{d}t = M(x,h). \end{aligned}$$

We have $m(x,h)\to f(x)$ and $M(x,h)\to f(x)$ as $h\to 0$. ■

Corollaries to Fundamental Theorem

Corollary: The derivative of an indefinite Riemann integral exists almost everywhere and equals the integrand almost everywhere.
Corollary: Every continuous function has an antiderivative.

Theorem: (Antiderivative theorem) Any antiderivative of a Riemann-integrable function, if it exists, differs from the indefinite integral by a constant.

Proof: Assume that $f : [a, b] \to \mathbb{R}$ is Riemann-integrable and suppose that G is an antiderivative of f (meaning $G'(x) = f(x)$). The theorem claims that there is a constant C so that

$$G(x) = \int_a^x f(t)\mathrm{d}t + C.$$

Define a partition P of $[a, x]$ by $a = x_0 < x_1 < \ldots < x_n = x$. By the Mean Value Theorem, we can pick $t_k \in [x_{k_1}, x_k]$ (definine a T so P, T is a partition pair) so that

$$G(x_k) - G(x_{k-1}) = G'(t_k)\Delta x_k = f(x)\Delta x_k.$$

Antiderivative theorem

Telescoping sum:

$$G(x) - \underbrace{G(a)}_{=C} = \sum_{k=1}^{n} G(x_k) - G(x_{k-1}) = \sum_{k=1}^{n} f(t_k)\Delta x_k = R(f, P, T).$$

As the mesh of P goes to zero, the right-hand side becomes $F(x)$, completing the proof. ■

Corollary: "Standard" integral formulas work as expected, i.e.

$$\int_a^b f'(x)\mathrm{d}x = f(b) - f(a).$$