Recall $\mathbb{N} = \{1, 2, 3, \ldots\}$. Definition (directly from book) of convergence of the sequence (a_n) to the real number b, i.e. the formal meaning of "$\lim_{n\to\infty} a_n = b$":

$$\forall\epsilon > 0 \exists N \in \mathbb{N}, \text{ such that } n > N \to (|a_n - b| < \epsilon).$$

In class, I wrote a slightly different, but equivalent thing (I was not careful to mention that $N \in \mathbb{N}$, but our author is similarly not careful to mention that $n \in \mathbb{N}$ in his definition, although that should be clear from its use in writing the sequence "a_n"):

$$\forall\epsilon > 0 \exists N > 0 \forall n > N(|a_n - b| < \epsilon).$$

Example 1. *Prove that* $\lim_{n\to\infty} \frac{n}{n+1} = 1$.

Proof. Let $\epsilon > 0$, let $N > \frac{1}{\epsilon} - 1$, and let $n > N$ (*note: if* $\epsilon > 1$, *then the requirement that* $N \in \mathbb{N}$ *means we can take, say,* $N = 1$). For such n, we have

$$n + 1 > N + 1 > \left(\frac{1}{\epsilon} - 1\right) + 1 = \frac{1}{\epsilon},$$

and thus $\frac{1}{n+1} < \epsilon$. Therefore, calculate

$$\left|\frac{n}{n+1} - 1\right| = \left|\frac{1}{1+n}\right| < \epsilon,$$

completing the proof. ■ □

Example 2. *Prove that* $\lim_{n\to\infty} \frac{2n}{3n+5} = \frac{2}{3}$.

Proof. Let $\epsilon > 0$, let $N > \frac{10}{9\epsilon} - \frac{5}{3}$, and let $n > N$. For such n, we have

$$3n + 5 > 3N + 5 > 3\left(\frac{10}{9\epsilon} - \frac{5}{3}\right) + 5 = \frac{10}{3\epsilon}.$$

Thus $\frac{9n+15}{10} > \frac{1}{\epsilon}$ and we have $\frac{10}{9n+15} < \epsilon$. Therefore, calculate

$$\begin{aligned} \left|\frac{2n}{3n+5} - \frac{2}{3}\right| &= \left|\frac{6n - (6n+10)}{9n+15}\right| \\ &= \left|\frac{-10}{9n+15}\right| \\ &= \frac{10}{9n+15} \\ &< \epsilon, \end{aligned}$$

completing the proof. ■ □