Pointwise and uniform convergence

Definition ("pointwise convergence"): Let $f_n \colon [a, b] \to \mathbb{R}$ be a sequence of functions. We say that f_n converges pointwise to f provided that $\forall x \in [a, b] \forall \epsilon > 0 \exists N \in \mathbb{N}$ so that for all $n \geq N, |f_n(x) - f(x)| < \epsilon$. For this, we write $f_n \to f$.

Definition ("uniform convergence"): Let $f_n \colon [a, b] \to \mathbb{R}$ be a sequence of functions. We say that f_n converges uniformly to f provided that $\forall \epsilon > 0 \exists N \in \mathbb{N}$ so that for all $n \geq N$ and $x \in [a, b]$, $|f_n(x) - f(x)| < \epsilon$. For this, we write $f_n \rightrightarrows f$.

Karl Weierstrass

Weierstrass is sometimes called the “father of analysis”. He formalized the definition of continuity, proved the intermediate value theorem, the Bolzano-Weierstrass theorem (essentially – that closed bounded subsets of $\mathbb{R}^n$ are compact), and the Weierstrass approximation theorem.

His doctoral students included, among many others, Georg Cantor, Georg Frobenius, and Hans von Mangoldt.

The space $C^0([a,b],\mathbb{R})$

We define the notation $C^0([a,b],\mathbb{R})$ to be the set of functions $f\colon [a,b] \to \mathbb{R}$ which are continuous.

We say that a set X is dense in $C^0([a,b],\mathbb{R})$ provided that for each $f \in C^0([a,b],\mathbb{R})$ and $\epsilon > 0$, there exists $p \in X$ so that for all $x \in [a,b]$,

$$|f(x) - p(x)| < \epsilon.$$

Weierstrass approximation theorem

Theorem: The set of polynomials is dense in the space of continuous functions.

Proof: Without loss of generality, assume $[a, b] = [0, 1]$. We may also assume that $f(0) = f(1) = 0$ because otherwise, the function

$$g(x) = f(x) - f(0) - x[f(1) - f(0)]$$

obeys $g(0) = g(1) = 0$ and if there is a sequence $f_n^* \rightrightarrows g$, then there is a sequence $f_n \rightrightarrows f$. We also define f to be identically zero outside of $[0, 1]$.

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$$Q_n(x) = c_n(1 - x^2)^n,$$

where c_n is chosen so that

$$c_n = \int_{-1}^{1} Q_n(x)\mathrm{d}x = 1.$$

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Calculate

$$\begin{aligned}\int_{-1}^{1}(1-x^2)^n \mathrm{d}x &= 2\int_{0}^{1}(1-x^2)^n \mathrm{d}x \\ &\geq 2\int_{0}^{\frac{1}{\sqrt{n}}}(1-x^2)^n \mathrm{d}x\end{aligned}$$

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Therefore

$$c_n < \sqrt{n}.$$

Weierstrass approximation theorem

For any $\delta > 0$, the fact that $c_n < \sqrt{n}$ implies for $\delta \leq |x| \leq 1$,

$$Q_n(x) = c_n(1 - x^2)^n \leq \sqrt{n}(1 - \delta^2)^n$$

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from which we conclude (HW) that $Q_n(x) \rightrightarrows 0$ on $[\delta, 1]$. Define for $x \in [0, 1]$,

$$P_n(x) = \int_{-1}^{1} f(x + t) Q_n(t) \mathrm{d}t.$$

Since f is identically zero outside of $[0, 1]$, it is identically zero whenever $x + t > 1$, i.e. when $t > 1 - x$ and it is identically zero when $x + t < 0$, i.e. $t < -x$.

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Letting $u = x + t$ so that $\mathrm{d}u = \mathrm{d}t$. Performing the change of variables gives us

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Letting $u = x + t$ so that $\mathrm{d}u = \mathrm{d}t$. Performing the change of variables gives us $t = u - x$, and if $t = -x$ then $u = 0$ and if $t = 1 - x$ then $u = x + (1 - x) = 1$, hence we get

$$P_n(x) = \int_{0}^{1} f(t)Q_n(t-x)\mathrm{d}t.$$

Note that this final integral is a polynomial in x. Therefore (P_n) is a sequence of polynomials.

Weierstrass approximation theorem

Let $\epsilon > 0$ and since f is continuous, we may choose $\delta > 0$ so that if $|y - x| < \delta$, then $|f(y) - f(x)| < \frac{\epsilon}{2}$. Let $M = \sup |f(x)|$, which exists because f is continuous on a compact interval. Compute

$$|P_n(x) - f(x)| \quad =$$

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$$\begin{aligned} |P_n(x) - f(x)| &= \left| \int_{-1}^{1} f(x+t) Q_n(t) \mathrm{d}t - f(x) \right| \\ &\overset{\int Q_n = 1}{=} \left| \int_{-1}^{1} [f(x+t) - f(x)] Q_n(t) \mathrm{d}t \right| \end{aligned}$$

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Weierstrass approximation theorem

Recall we chose δ so if $|y-x|<\delta$, then $|f(y)-f(x)|<\frac{\epsilon}{2}$ and we defined the quantity $M=\sup|f(x)|$. We also had $Q_n(x)\leq\sqrt{n}(1-\delta^2)^n$ for $\delta\leq|x|\leq 1$.

$$|P_n(x)-f(x)| \leq \int_{-1}^{1}[|f(x+t)-f(x)|]Q_n(t)\mathrm{d}t$$

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\begin{aligned}
|P_n(x) - f(x)| &\leq \int_{-1}^{1} [|f(x+t) - f(x)|] Q_n(t) \mathrm{d}t \\
&= \left[\int_{-1}^{-\delta} + \int_{-\delta}^{\delta} + \int_{\delta}^{1} \right] [|f(x+t) - f(x)|] Q_n(t) \mathrm{d}t
\end{aligned}
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&\le 2M\sqrt{n}(1-\delta^2)^n\left[\int_{-1}^{0}+\int_{0}^{1}\right]1\mathrm{d}t+\frac{\epsilon}{2}\int_{-1}^{1}Q_n(t)\mathrm{d}t\\
&= 4M\sqrt{n}(1-\delta^2)^n+\frac{\epsilon}{2}\\
&\to \frac{\epsilon}{2}\\
&< \epsilon. \blacksquare
\end{aligned}
$$

Bernstein polynomials

If $f\colon [0,1] \to \mathbb{R}$, then the Bernstein polynomials $B_n(f;x)$ defined by

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}$$

converge to f uniformly on $[0,1]$.

Quiz: Calculate and plot the first 5 Bernstein polynomials for $f(x) = \sin(2\pi x)e^{-x^2}$.