Pointwise and uniform convergence

Definition ("pointwise convergence"): Let $f_n\colon [a,b] \to \mathbb{R}$ be a sequence of functions. We say that f_n converges pointwise to f provided that $\forall x \in [a,b] \forall \epsilon > 0 \exists N \in \mathbb{N}$ so that for all $n \geq N, |f_n(x) - f(x)| < \epsilon$. For this, we write $f_n \to f$.

Definition ("uniform convergence"): Let $f_n\colon [a,b] \to \mathbb{R}$ be a sequence of functions. We say that f_n converges uniformly to f provided that $\forall \epsilon > 0 \exists N \in \mathbb{N}$ so that for all $n \geq N$ and $x \in [a,b]$, $|f_n(x) - f(x)| < \epsilon$. For this, we write $f_n \rightrightarrows f$.

Mean value theorem, Cauchy Criterion, and limit theorem

Theorem: (Mean Value Theorem) Let $f\colon [a, b] \to \mathbb{R}$ be continuous and differentiable on (a, b). Then there exists $\theta \in (a, b)$ so that

$$\frac{f(b) - f(a)}{b - a} = f'(\theta).$$

Theorem: Suppose that $f_n\colon [a, b] \to \mathbb{R}$. Then $f_n \rightrightarrows f$ if and only if $\forall \epsilon > 0 \exists N \in \mathbb{N}$ such that for all $m, n \geq N$ and for all $x \in [a, b]$, $|f_n(x) - f_m(x)| < \epsilon$.

Theorem: Suppose that $f_n\colon [a, b] \to \mathbb{R}$, that $f_n \rightrightarrows f$, let $x \in [a, b]$, and suppose that $\lim\limits_{t \to x} f_n(t) = A_n$. Then A_n is a convergent sequence and

$$\lim_{t \to x} f(t) = \lim_{n \to \infty} A_n.$$

Another example

Let $f_n \colon \mathbb{R} \to \mathbb{R}$ with $f_n(x) = \dfrac{\sin(nx)}{\sqrt{n}}$.

1. Compute $\lim_{n\to\infty} f_n(x)$ (call this function $\mathcal{F}(x)$). Ans: $\mathcal{F}(x) = 0$
2. Is that convergence uniform? Ans: Yes!
3. Compute $\mathcal{F}'(x)$. Ans: 0
4. Compute $f_n'(x)$. Ans: $f_n'(x) = \sqrt{n}\cos(nx)$
5. Compute $\lim_{n\to\infty} f_n'(x)$. Ans: ∞, $-\infty$, or undefined depending on the sign of $\cos(nx)$ For instance, if $x = 0$ you get $\lim_{n\to\infty} \sqrt{n} = \infty$.

   Consequence: Even the uniform limit of differentiable functions can fail to be differentiable!

Uniform convergence and differentiation

Theorem: Suppose $f_n \colon [a, b] \to \mathbb{R}$ is a sequence of differentiable functions for which $f_n(x_0)$ converges. If f_n' converges uniformly on $[a, b]$, then f_n converges uniformly on $[a, b]$ to a function f and

$$f'(x) = \lim_{n\to\infty} f_n'(x).$$

Proof: Let $\epsilon > 0$. Since $f_n(x_0)$ converges, choose $N \in \mathbb{N}$ so that if $n, m \geq N$ implies

$$|f_n(x_0) - f_m(x_0)| < \frac{\epsilon}{2}.$$

Since f_n' converges uniformly, the N earlier can be chosen large enough so that for **all** $t \in [a, b]$,

$$|f_n'(t) - f_m'(t)| < \frac{\epsilon}{2(b-a)}.$$

Uniform convergence and differentiation

Since f_n and f_m are differentiable, $f_n - f_m$ is differentiable. Let $x, t \in [a, b]$ be any two points (without loss of generality, assume $t < x$). By the mean value theorem applied to the function $g(x) = f_n(x) - f_m(x)$, there exists $\theta \in [a, b]$ so that

$$\frac{g(x) - g(t)}{x - t} = g'(\theta),$$

in other words and taking absolute value,

$$\frac{|(f_n(x) - f_m(x)) - (f_n(t) - f_m(t))|}{|x - t|} = |f_n'(\theta) - f_m'(\theta)|.$$

So we have: for **all** $t \in [a, b]$, $|f_n'(t) - f_m'(t)| < \frac{\epsilon}{2(b-a)}$ and

$$|(f_n(x) - f_m(x)) - (f_n(t) - f_m(t))| = |x-t||f_n'(\theta)-f_m'(\theta)| < \frac{|x-t|\epsilon}{2(b-a)} < \frac{\epsilon}{2}.$$

Compute (how?)

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f_m(x) - f_n(x_0) + f_m(x_0)| + |f_n(x_0) - f_m(x_0)|.$$

Uniform convergence and differentiation

We have

$$|f_n(x_0) - f_m(x_0)| < \frac{\epsilon}{2},$$

and for **all** $x, t \in [a, b]$, $|(f_n(x) - f_m(x)) - (f_n(t) - f_m(t))| < \frac{\epsilon}{2}$ and

$$|f_n(x) - f_m(x)| \leq \underbrace{|f_n(x) - f_m(x) - f_n(x_0) + f_m(x_0)|}_{<\frac{\epsilon}{2}} + \underbrace{|f_n(x_0) - f_m(x_0)|}_{<\frac{\epsilon}{2}}.$$

Therefore we have for any $x \in [a, b]$ and any $m, n \geq N$,

$$|f_n(x) - f_m(x)| < \epsilon.$$

Therefore by the Cauchy criterion, we may conclude that f_n converges uniformly on $[a, b]$.

Uniform convergence and differentiation

Define the limit function $f\colon [a,b] \to \mathbb{R}$ by $f(x) = \lim_{n\to\infty} f_n(x)$. Fix $x \in [a,b]$ and define two functions $\phi_n, \phi\colon [a,b] \setminus \{x\} \to \mathbb{R}$:

$$\phi_n(t) = \frac{f_n(t) - f_n(x)}{t - x}, \quad \phi(t) = \frac{f(t) - f(x)}{t - x}.$$

By definition of differentiation, for any $n = 1, 2, 3, \dots$ we get

$$\lim_{t\to x} \phi_n(t) = f_n'(x).$$

Uniform convergence and differentiation

In the new notation, we may rewrite the inequality (which held for **all** $x, t \in [a, b]$)

$$|(f_n(x) - f_m(x)) - (f_n(t) - f_m(t))| < \frac{|x - t|\epsilon}{2(b - a)}$$

by dividing by $|x - t|$, we get

$$\left|\frac{(f_m(t) - f_m(x)) - (f_n(t) - f_n(x))}{|x - t|}\right| = |\phi_m(t) - \phi_n(t)| < \frac{\epsilon}{2(b - a)}.$$

This establishes by the Cauchy criterion that ϕ_n converges uniformly whenever $t \in [a, b] \setminus \{x\}$.

Uniform convergence and differentiation

Since ϕ_n converges uniformly and we know that $f_n \rightrightarrows f$, we can say by the definitions of ϕ_n and ϕ that

$$\lim_{n\to\infty} \phi_n(t) = \phi(t).$$

Therefore since we also know

$$\lim_{t\to x} \phi_n(t) = f_n'(x),$$

by the limit theorem,

$$\lim_{t\to x} \phi(t) = \lim_{n\to\infty} f_n'(x).$$

So, by definition of ϕ we conclude

$$f'(x) = \lim_{n\to\infty} f_n'(x). \blacksquare$$