# Pointwise and uniform convergence

**Definition** ("pointwise convergence"): Let $f_n \colon [a, b] \to \mathbb{R}$ be a sequence of functions. We say that $f_n$ converges pointwise to $f$ provided that $\forall x \in [a, b] \forall \epsilon > 0 \exists N \in \mathbb{N}$ so that for all $n \geq N, |f_n(x) - f(x)| < \epsilon$. For this, we write $f_n \to f$.

**Definition** ("uniform convergence"): Let $f_n \colon [a, b] \to \mathbb{R}$ be a sequence of functions. We say that $f_n$ converges uniformly to $f$ provided that $\forall \epsilon > 0 \exists N \in \mathbb{N}$ so that for all $n \geq N$ and $x \in [a, b]$, $|f_n(x) - f(x)| < \epsilon$. For this, we write $f_n \rightrightarrows f$.

# Cauchy sequence-like theorem

**Theorem**: Suppose that $f_n \colon [a, b] \to \mathbb{R}$. Then $f_n \rightrightarrows f$ if and only if $\forall \epsilon > 0 \exists N \in \mathbb{N}$ such that for all $m, n \geq N$ and for all $x \in [a, b]$, $|f_n(x) - f_m(x)| < \epsilon$.

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*Proof*: $(\rightarrow)$ Let $\epsilon > 0$. Suppose that $f_n \rightrightarrows f$. Then there is $N \in \mathbb{N}$ so that if $n, m \geq N$ and $x \in [a, b]$, then

$$|f_n(x) - f(x)| < \frac{\epsilon}{2}$$

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Therefore

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f_m(x) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

($\leftarrow$) Let $\epsilon > 0$. By assumption, there is $N \in \mathbb{N}$ so that for all $m, n \geq N$ and $x \in [a, b]$,

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($\leftarrow$) Let $\epsilon > 0$. By assumption, there is $N \in \mathbb{N}$ so that for all $m, n \geq N$ and $x \in [a, b]$,

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Since $f_m(x)$ is a sequence in $\mathbb{R}$ that converges (why: we showed "being a convergent sequence" is the same as "being a Cauchy sequence" a long time ago) to some $f(x)$, we take the limit as $m \to \infty$ of the above expression which yields

$$|f_n(x) - f(x)| < \epsilon. \blacksquare$$

# Properties of uniform convergence

**Theorem**: Suppose that $f_n \colon [a,b] \to \mathbb{R}$, that $f_n \rightrightarrows f$, let $x \in [a,b]$, and suppose that $\lim_{t \to x} f_n(t) = A_n$. Then $A_n$ is a convergent sequence and

$$\lim_{t \to x} f(t) = \lim_{n \to \infty} A_n.$$

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*Proof*: Let $\epsilon > 0$. Since $f_n \rightrightarrows f$, there exists $N \in \mathbb{N}$ so that for all $n, m \geq N$ and for all $t \in [a, b]$,

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Taking the limit as $t \to x$, we get

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**Goal:** show that there is large enough $n$ and $t \in [a,b]$ close enough to $x$ (i.e. find a "$\delta$") so that

$$|f(t) - A_n| < \epsilon.$$

So, since $f_n \rightrightarrows f$, we may choose $n$ large enough so that for all $t \in [a, b]$,

$$|f(t) - f_n(t)| < \frac{\epsilon}{3}$$

and

$$|A_n - A| < \frac{\epsilon}{3}.$$

Since $\lim_{t \to x} f_n(t) = A_n$, we may pick a $\delta > 0$ so that if $|t - x| < \delta$, then $|f_n(t) - A_n| < \frac{\epsilon}{3}$.

Therefore, for $t$ so that $|t - x| < \delta$,

$$\begin{aligned} |f(t) - A| &\leq |f(t) - f_n(t)| + |f_n(t) - A| \\ &\leq |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A| \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &< \epsilon, \end{aligned}$$

completing the proof. ■

**Corollary**: The uniform limit of continuous functions is continuous.

# Properties of uniform convergence

**Theorem**: Suppose that $f_n\colon [a,b] \to \mathbb{R}$ are Riemann-integrable and suppose that $f_n \rightrightarrows f$. Then $f$ is Riemann-integrable and

$$\int_a^b f(x)\mathrm{d}x = \lim_{n\to\infty} \int f_n(x)\mathrm{d}x.$$

*Proof*: Define $\varepsilon_n = \sup_{x\in[a,b]} |f_n(x) - f(x)|$. Then for all $x \in [a,b]$,

$$f_n(x) - \varepsilon_n \leq f(x) \leq f_n(x) + \varepsilon_n.$$

(note 1: we know $\lim \varepsilon_n = 0$)
(note 2: we don't know that $f$ is integrable yet! So we need to argue using lower and upper sums)

$$\int_a^b f_n(x) - \varepsilon_n \mathrm{d}x \leq \underline{\int} f(x)\mathrm{d}x \leq \overline{\int} f(x)\mathrm{d}x \leq \int_a^b f_n(x) + \varepsilon_n \mathrm{d}x.$$

# Properties of uniform convergence

$$\int_a^b f_n(x) - \varepsilon_n \mathrm{d}x \leq \underline{\int} f(x)\mathrm{d}x \leq \overline{\int} f(x)\mathrm{d}x \leq \int_a^b f_n(x) + \varepsilon_n \mathrm{d}x.$$

Thus since $\int_a^b f_n(x) - \varepsilon_n \mathrm{d}x \leq \underline{\int} f(x)\mathrm{d}x$ implies

$$-\underline{\int} f(x)\mathrm{d}x \leq -\int_a^b f_n(x) - \varepsilon_n \mathrm{d}x,$$

we see

$$\begin{aligned} 0 \leq \overline{\int} f(x)\mathrm{d}x - \underline{\int} f(x)\mathrm{d}x &\leq \int_a^b f_n(x) + \varepsilon_n \mathrm{d}x - \underline{\int} f(x)\mathrm{d}x \\ &\leq \int_a^b f_n(x) + \varepsilon_n \mathrm{d}x - \int_a^b f_n(x) - \varepsilon_n \mathrm{d}x = 2\varepsilon_n(b-a), \end{aligned}$$

taking the limit as $n \to \infty$ completes the proof. ■

# Another example

Let $f_n \colon \mathbb{R} \to \mathbb{R}$ with $f_n(x) = \dfrac{\sin(nx)}{\sqrt{n}}$.

1. Compute $\lim_{n\to\infty} f_n(x)$ (call this function $\mathcal{F}(x)$).
2. Is that convergence uniform?
3. Compute $\mathcal{F}'(x)$.
4. Compute $f_n'(x)$.
5. Compute $\lim_{n\to\infty} f_n'(x)$.

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Let $f_n \colon \mathbb{R} \to \mathbb{R}$ with $f_n(x) = \dfrac{\sin(nx)}{\sqrt{n}}$.

1. Compute $\lim_{n\to\infty} f_n(x)$ (call this function $\mathcal{F}(x)$).Ans: $\mathcal{F}(x) = 0$
2. Is that convergence uniform? Ans: Yes!
3. Compute $\mathcal{F}'(x)$. Ans: 0
4. Compute $f_n'(x)$. $f_n'(x) = \sqrt{n}\cos(nx)$
5. Compute $\lim_{n\to\infty} f_n'(x)$. Ans: $\infty$, $-\infty$, or undefined depending on the sign of $\cos(nx)$ For instance, if $x = 0$ you get $\lim_{n\to\infty} \sqrt{n} = \infty$.

   **Consequence**: Even the uniform limit of differentiable functions can fail to be differentiable!