Homework 9 — MATH 4590 Spring 2018

1. Use induction to prove that

$$1 + r + r^2 + \ldots + r^n = \frac{1 - r^{n+1}}{1 - r}.$$

*Proof*: The case $n = 1$ says $1 + r = \frac{1 - r^2}{1 - r}$. Since $1 - r^2 = (1 - r)(1 + r)$ we get $\frac{(1 - r)(1 + r)}{1 - r} = 1 + r$, completing this case. Now assume it holds for $n = N$, i.e. assume

$$(*) \qquad 1 + r + r^2 + \ldots + r^N = \frac{1 - r^{N+1}}{1 - r}.$$

We now want to show it holds, i.e.

$$\textbf{Goal: } 1 + r + r^2 + \ldots + r^N + r^{N+1} = \frac{1 - r^{N+2}}{1 - r}.$$

Start with the left-hand side:

$$\begin{aligned} 1 + r + \ldots + r^N + r^{N+1} &\overset{(*)}{=} \frac{1 - r^{N+1}}{1 - r} + r^{N+1} \\ &= \frac{1 - r^{N+1}}{1 - r} + \frac{r^{N+1} - r^{N+2}}{1 - r} \\ &= \frac{1 - r^{N+2}}{1 - r}, \end{aligned}$$

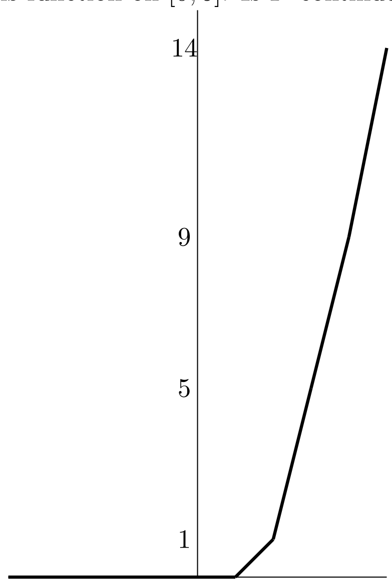
completing the proof. ■

2. Define

$$F(x) = \begin{cases} 0, & x < 0 \\ (n-1)x - \dfrac{(n-1)n}{2}, & x \in [n-1, n), n \in \{1, 2, 3, \ldots\} \end{cases}$$

a.) Sketch this function on $[0, 5]$. Is $F$ continuous?

*Solution*:




b.) Find $F'(x)$ at places which have a derivative. Add a sketch for $F'$ to your sketch in part a.).

*Solution*:

$$F'(x) = \begin{cases} 0, & x < 1 \\ 1, & 1 < x < 2 \\ 2, & 2 < x < 3 \\ 3, & 3 < x < 4 \\ \vdots & \vdots \\ n, & n < x < n+1 \\ \vdots & \vdots \end{cases}$$

Which is identical to $\lfloor x \rfloor$ for any $x \notin \mathbb{Z}$. (note: it is ok to not equal $\lfloor x \rfloor$ at $x \in \mathbb{Z}$ because $\mathbb{Z}$ is a zero set and so it does not contribute to the integral in part c)!!)

c.) Use the above part to evaluate $\int_a^b \lfloor x \rfloor \mathrm{d}x$, where $\lfloor x \rfloor$ denotes the floor function (i.e. $\lfloor x \rfloor$ is the greatest integer $\leq x$)

*Solution*: We get, since $\lfloor x \rfloor \overset{\text{almost everywhere}}{=} F'(x)$,

$$\int_a^b \lfloor x \rfloor \mathrm{d}x = \int_a^b F'(x) \mathrm{d}x = F(b) - F(a),$$

where $F$ is the function defined in this question.

3. Prove that if $Z_1$ and $Z_2$ are zero sets, then $Z_1 \cup Z_2$ is a zero set.
*NOTE*: Recall that a set $Z$ is called a zero set provided that for any $\epsilon > 0$ there exists a sequence of interval $(\alpha_n, \beta_n)$ so that for any $\xi \in Z$, $\xi \in (\alpha_j, \beta_j)$ for some $j$ (this means the intervals "cover" $Z$) and the sum of the lengths of these intervals is $< \epsilon$, i.e.

$$\sum_{n=1}^{\infty} \text{length}\Big((\alpha_n, \beta_n)\Big) = \sum_{n=1}^{\infty} \beta_n - \alpha_n < \epsilon.$$

*Proof*: Suppose that $Z_1$ and $Z_2$ are zero sets. Let $\epsilon > 0$. We need to show that $Z_1 \cup Z_2$ is also a zero set. Since $Z_1$ is a zero set, there is a sequence of intervals $(\alpha_n, \beta_n)$ covering $Z_1$ so that

$$\sum_{n=1}^{\infty} \beta_n - \alpha_n < \frac{\epsilon}{2}.$$

Similarly, there is a different sequence of intervals $(\gamma_n, \delta_n)$ that cover $Z_2$ so that

$$\sum_{n=1}^{\infty} \delta_n - \gamma_n < \frac{\epsilon}{2}.$$

Define a new sequence of intervals

$$(\phi_n, \psi_n) = \begin{cases} (\alpha_n, \beta_n), & n = 1, 3, 5, 7, \ldots \\ (\gamma_n, \delta_n), & n = 2, 4, 6, 8, \ldots. \end{cases}$$

Then we observe that the sequence $(\phi_n, \psi_n)$ covers $Z_1 \cup Z_2$ and

$$\sum_{n=1}^{\infty} \psi_n - \phi_n = \left( \sum_{n \in \{1,3,5,7,\ldots\}} \beta_n - \alpha_n \right) + \left( \sum_{n \in \{2,4,6,8,\ldots\}} \delta_n - \gamma_n \right) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

proving that $Z_1 \cup Z_2$ is a zero set. ■

4. Show that any two antiderivatives of a function $f$ differ by a constant.
(*hint: use the "antiderivative theorem"*)
*Solution*: Let $F_1$ and $F_2$ be any two antiderivatives of $f$. By the antiderivative theorem, each of these differ from the indefinite integral $I = \displaystyle\int_a^x f(t)\mathrm{d}t$ by a constant, say $F_1 = I + C_1$ and $F_2 = I + C_2$. Now we see that

$$F_1 - F_2 = (I + C_1) - (I + C_2) = C_1 - C_2,$$

or in other words, $F_1$ and $F_2$ differ by a constant, as was to be shown.

5. If $f_n \rightrightarrows f$ and $g_n \rightrightarrows g$ on $A$, prove that $f_n + g_n \rightrightarrows f + g$ on $A$.
*Solution*: Let $\epsilon > 0$. Since $f_n \rightrightarrows f$, we know there exists $N_f$ so that for all $n \geq N_f$ and $\forall x \in A$,

$$|f_n(x) - f(x)| < \frac{\epsilon}{2}.$$

Similarly, since $g_n \rightrightarrows g$, we know there exists $N_g$ so that for all $n \geq N_g$ and $\forall x \in A$,

$$|g_n(x) - g(x)| < \frac{\epsilon}{2}.$$

Therefore if we choose $N = \max\{N_f, N_g\}$ and let $n \geq N$, we may calculate for all $x \in A$,

$$\begin{aligned} |(f_n(x) + g_n(x)) - (f(x) + g(x))| &= |(f_n(x) - f(x)) + (g_n(x) - g(x))| \\ &\leq |f_n(x) - f(x)| + |g_n(x) - g(x)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon, \end{aligned}$$

completing the proof. ■