Homework 7 — MATH 4590 Spring 2018

1. Assume that $f\colon \mathbb{R} \to \mathbb{R}$ satisfies $|f(t) - f(x)| \leq |t - x|^2$ for all $t, x \in \mathbb{R}$. Prove that f is a constant function.

2. (a) Draw the graph of a continuous function defined on $[0, 1]$ that is differentiable on $(0, 1)$, but not at the endpoints.
   (*hint: take inspiration from the graph of the following function, which is continuous on $\mathbb{R}$ but not differentiable at zero:*
   $f(x) = \left\{ \begin{array}{ll} 0, & x = 0 \\ x \sin\left(\frac{1}{x}\right), & x \neq 0 \end{array} \right\}$)

   (b) Can you find a formula for such a function? (not necessarily the one you drew)

   (c) Does the Mean Value Theorem apply to such a function? Why or why not?

3. Assume that the functions f and g are smooth (i.e. infinitely differentiable). Prove the Leibniz product rule: for any $r \in \{1, 2, 3, \ldots\}$,

$$(f \cdot g)^r (x) = \sum_{k=0}^{r} \binom{r}{k} f^{(k)}(x) g^{r-k}(x),$$

   where $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$.
   (*hint: use induction*)

4. Recall the rth order Taylor polynomial of a r-times differentiable function $f(x)$ is given by

$$P(h) = \sum_{k=0}^{r} \frac{f^{(k)}(x)}{k!} h^k.$$

   Also recall that the remainder $R(h)$ is given by

$$R(h) = f(x + h) - P(h),$$

   and (assuing f is $r + 1$-times differentiable) that the Taylor Approximation Theorem (part c) says that there is some $\theta \in (0, h)$ so that

$$R(h) = \frac{f^{(r+1)}(\theta)}{(r + 1)!} h^{r+1}.$$

   We will investigate the following question: what is the max possible error that a 5th order Taylor polynomial for the function $f(x) = \sin(x)$ centered at zero may have in approximating $\sin\left(\frac{1}{2}\right)$?

(a) Find the 5th order Taylor polynomial of $f(x) = \sin(x)$.

(b) Set $x = 0$ in the resulting formula from above (this "centers" the Taylor polynomial near x).

(c) Use the Taylor Approximation Theorem (part c) to write a formula (in terms of θ) for $R\left(\frac{1}{2}\right)$ (with $x = 0$).

(d) Find a number $\xi \in \mathbb{R}$ such that $\left|R\left(\frac{1}{2}\right)\right| < \xi$, where ξ does not depend on θ (with $x = 0$).

5. Define $f(x) = \begin{cases} x^2, & x < 0 \\ x + x^2, & x \geq 0. \end{cases}$
   Differentiation gives $f''(x) = 2$. This is bogus. Why? (*hint: draw a picture of f and its derivative*)