Homework 6 — MATH 4590 Spring 2018

1.) Chapter 3, # 6 from the text: If $f\colon (a,b) \to \mathbb{R}$ assumes a maximum or minimum at some $\theta \in (a,b)$, prove that $f'(\theta) = 0$.
Proof: We need to show that $f'(\theta) = 0$, i.e. that

$$f'(\theta) \stackrel{\text{def}}{=} \lim_{t\to\theta} \frac{f(t) - f(\theta)}{t - \theta} = 0.$$

Assume without loss of generality that θ is the location of a local minimum (i.e. there is a δ such that for all $t \in (\theta - \delta, \theta + \delta)$, $f(\theta) \le f(t)$. Hence for all such t, $f(t) - f(\theta) \ge 0$. Assume that $t \in (\theta - \delta, \theta)$, which means that $t < \theta$. Then $t - \theta < 0$ and so

$$\frac{f(t) - f(\theta)}{t - \theta} < 0.$$

On the other hand, if $t \in (\theta, \theta + \delta)$, then $t - \theta > 0$ and so

$$\frac{f(t) - f(\theta)}{t - \theta} > 0.$$

Combining these formulas yields the following for $t \in (\theta - \delta, \theta + \delta)$:

$$0 < \frac{f(t) - f(\theta)}{t - \theta} < 0.$$

Taking the limit as $t \to \theta$ and using the fact that the limit in the middle exists since f is differentiable at θ forces us to conclude that

$$0 \le \lim_{t\to\theta} \frac{f(t) - f(\theta)}{t - \theta} = f'(\theta) \le 0,$$

in other words $f'(\theta) = 0$, as was to be shown. The proof when θ is a local maximum is very similar. ■

2.) Prove using induction and the rules of differentiation (or directly, if you want) that the derivative of x^n is nx^{n-1}.
Solution: The case $n = 0$ is the formula $\frac{\mathrm{d}}{\mathrm{d}x} x = 1$ which we prove:

$$\lim_{t\to x} \frac{x - t}{x - t} = \lim_{t\to x} 1 = 1 = 1x^{1-1}.$$

Now assume the formula holds for $n = N - 1$. We now prove it for $n = N$:

$$\frac{\mathrm{d}}{\mathrm{d}x} x^N = \frac{\mathrm{d}}{\mathrm{d}x} x \cdot x^{N-1} \stackrel{\text{prod rule}}{=} 1 \cdot x^{N-1} + (N-1)x \cdot x^{N-2} = (N)x^{N-1},$$

completing the proof.

3.) The notion of an "absolute derivative" in metric spaces was recently defined: let (M, d_1) and (N, d_2) be metric spaces and let $f\colon M \to N$ be a function. The absolute derivative of f, $f^{|'|}\colon M \to [0, \infty)$, is defined by

$$f^{|'|}(x) = \lim_{t \to x} \frac{d_2(f(x), f(t))}{d_1(x,t)}.$$

Fix a point $y \in N$ and let f be the constant function $f(x) = y$. Prove that $f^{|'|}(x) = 0$.
Solution: Since f is the constant function $f(x)y$, compute

$$f^{|'|}(x) = \lim_{t \to x} \frac{d_2(f(x), f(t))}{d_1(x,t)} = \lim_{t \to x} \frac{d_2(y,y)}{d_2(x,t)} = \lim_{t \to x} 0 = 0.$$

4.) Prove that if $f\colon \mathbb{R} \to \mathbb{R}$ (taking the usual metric in each case) is differentiable, then $f^{|'|}(x) = |f'(x)|$. $\Big($*hint: you may find the fact that, for a function g,* $\lim\limits_{t \to x} |g(x)| = \left|\lim\limits_{t \to x} g(x)\right|$ *useful*$\Big)$
Solution: Compute

$$\begin{aligned} f^{|'|}(x) &= \lim_{t \to x} \frac{d_2(f(x), f(t))}{d_1(x,t)} \\ &\overset{\text{usual metric}}{=} \lim_{t \to x} \left| \frac{f(x) - f(t)}{x - t} \right| \\ &\overset{hint}{=} \left| \lim_{t \to x} \frac{f(x) - f(t)}{x - t} \right| \\ &= \left| f'(x) \right|. \end{aligned}$$