1. Let (M, d) be a metric space and let $U \subset M$. Prove that U is open if and only if none of its points are a limit of its complement.
Proof: ($\longrightarrow$) We need to prove "if U is open, then none of its points are a limit of U^c." Assume that U is open. We must show that none of its points are a limit of its complement. Assume that $p \in U$ is the limit of a sequence (p_n) lying in U^c. This means that $\forall \epsilon > 0 \exists N \in \mathbb{N}$ such that $n \geq N$ implies $d(p_n, p) < \epsilon$. Since U is open, we know there exists some $r > 0$ such that the set of all points q that are a distance less than r from p all lie in U, i.e. $B = \{q \in M : d(p, q) < r\} \subset U$. However, if we take $\epsilon = r$, the convergence of (p_n) implies that there is a $N \in \mathbb{N}$ so that for all $n \geq N$, $d(p_n, p) < \epsilon = r$. By definition of B, this means that $p_n \in B$. But this is a contradiction, because $B \subset U$ while $p_n \in B$ and $p_n \in U^c$. Therefore no such sequence (p_n) can exist.

($\longleftarrow$) We need to prove "if none of the points of U are a limit of U^c, then U is open". Suppose that U is not open. Then there is a point $p \in U$ such that for **any** $r > 0$, the collection $\{q \in M : d(p, q) < r\} \not\subset U$, from which we may conclude that $\{q \in M : d(p, q) < r\} \cap U^c \neq \emptyset$. Let $r_n = \frac{1}{n}$ and choose any $p_n \in \{q \in M : d(p, q) < r_n\} \cap U^c$. This gives us a sequence (p_n) in U^c. Notice in particular, that $d(p_n, p) < \frac{1}{n}$.

Claim: (p_n) converges to p

Proof of claim: Let $\epsilon > 0$ choose $N \in \mathbb{N}$ with $N > \frac{1}{\epsilon}$. Then for any $n \geq N$, we have $\frac{1}{n} \leq \frac{1}{N} < \epsilon$, so calculate

$$d(p_n, p) < \frac{1}{n} < \epsilon.$$

completing the proof of the claim.

But this is a contradiction to our hypothesis, because (p_n) is a sequence lying in U^c that converges to p. In other words, $p \in U$ is a limit of U^c. Therefore our assumption that U was not open is invalid, hence U is open, completing the proof. ■

2. Prove that the union of finitely many closed sets is a closed set.
Proof: Let $n \in \{1, 2, 3, \ldots\}$ be fixed and let $C_1, C_2, \ldots, C_n$ be closed sets in a metric space (M, d). We need to show that $\mathcal{C} = \bigcup_{k=1}^{n} C_k$ is closed. Consider the sets $C_1^c, C_2^c, \ldots C_n^c$. By Theorem 4 in Chapter 2 (complement of open set is closed set and complement of closed set is open set), we know that $C_1^c, \ldots, C_n^c$ are open sets. By Theorem 5 in Chapter 2 (set of all open subsets of M is a topology), we know that any intersection of finitely many open sets is an open set. Therefore we know that $\bigcap_{k=1}^{n} C_k^c$ is an open set. Again by Theorem 5, we know that $\left(\bigcap_{k=1}^{n} C_k^c\right)^c$ is a closed set. By deMorgan's law for sets and the fact that that complements cancel each other, we compute

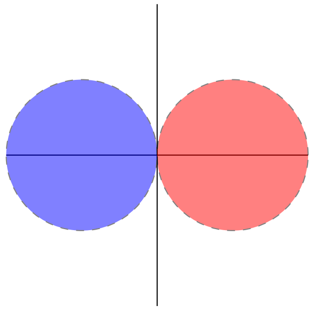
$$\left(\bigcap_{k=1}^{n} C_k^c\right)^c \overset{\text{deMorgan}}{=} \bigcup_{k=1}^{n} (C_k^c)^c = \bigcup_{k=1}^{n} C_k.$$

So, since we knew already that $\left(\bigcap_{k=1}^{n} C_k^c\right)^c$ was a closed set and we just saw that $\left(\bigcap_{k=1}^{n} C_k^c\right)^c = \bigcup_{k=1}^{n} C_k$, we must conclude that $\bigcup_{k=1}^{n} C_k$ is a closed set. In other words, the union of finitely many closed sets is a closed set, completing the proof. ■

3. Consider $(M, d) = (\mathbb{R}^2, d)$, where d is the Euclidean metric. We define the distance between two nonempty sets $A, B \subset \mathbb{R}^2$ by the following:

$$\text{dist}(A, B) = \inf\{d(a, b) \colon a \in A, b \in B\}.$$

Give an example of two nonempty *disjoint* sets $A, B \subset \mathbb{R}^2$ such that $\text{dist}(A, B) = 0$.



Solution:
Call the blue disk A and the red disk B. Notice that the boundaries of A and B are not included (this is necessary to satisfy $A \cap B = \emptyset$, i.e. that A and B are disjoint). We claim that $\text{dist}(A, B) = 0$. To see this, let (a_n) in A be the sequence $a_n = \left(-\frac{1}{n}, 0\right)$ and let $(b_n) = \left(\frac{1}{n}, 0\right)$. Then,

$$d(a_n, b_n) = \sqrt{\left(-\frac{1}{n} - \frac{1}{n}\right)^2 - (0-0)^2} = \sqrt{\frac{4}{n^2}} = \frac{2}{n}.$$

Since the limit as $n \to \infty$ of this distance is zero, we conclude that $\inf\{d(a_n, b_n) \colon n = 1, 2, 3, \ldots\} = 0$ and hence $\text{dist}(A, B) = 0$.