Homework 4 — MATH 4590 Spring 2018

Let (M, d) be a metric space and let $S \subset M$. A point $p \in M$ is called a limit of S provided that there exists a sequence (p_n) in S such that $p_n \to p$. We say that S is closed if it contains all of its limits. We say that S is open if $\forall p \in S \exists r > 0$ such that if $d(p, q) < r$, then $q \in S$ (in other words: you can make a small enough "ball" around p that all points in the ball are also in S).

1. Let $(M, d) = (\mathbb{R}^2, d)$, where d is the Euclidean metric

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Is the set $\mathbb{R} = \{(x, 0) \colon x \in \mathbb{R}\} \subset \mathbb{R}^2\}$ a closed, an open, both closed and open, or neither a closed nor an open subset of $\mathbb{R}^2$? Why or why not? (*hint: the set of points a certain distance away from a point in* $\mathbb{R}^2$*, in this metric, looks like a little disk (not including its boundary)*)

Solution: It is a closed set and it is not an open set. It is closed because given any converging sequence p_n in $\mathbb{R}$ where $p_n \to p$, the point p lives in $\mathbb{R}$:

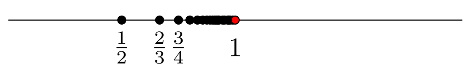


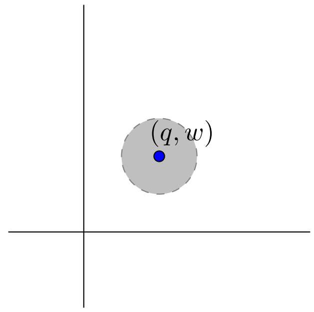
Figure 1: The sequence $1 - \dfrac{1}{n}$ in $\mathbb{R}$, which converges to 1.

That fact can be proven:

Theorem: If $p_n = (x_n, 0)$ is a convergent sequence in $(\mathbb{R}^2, d)$, where d is the Euclidean metric, then its limit is of the form $(q, 0)$ (i.e. it lies in $\mathbb{R} \subset \mathbb{R}^2$).

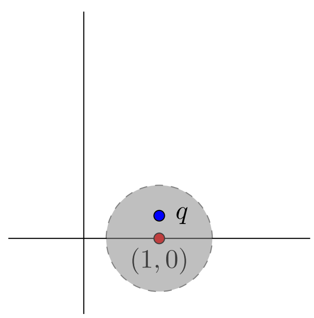
Proof: Suppose that the limit was not in $\mathbb{R} \subset \mathbb{R}^2$, then the limit would be of the form (q, w) with $w \neq 0$. Note that the ball of radius $\dfrac{|w|}{2}$ does not touch the x-axis, i.e. the points

$$M_{\frac{w}{2}}((q, w)) \cap \mathbb{R} = \left\{(a, b) \in \mathbb{R}^2 \colon d((q, w), (a, b)) < \frac{w}{2}\right\} \cap \mathbb{R} = \emptyset.$$



Choose $\epsilon = \dfrac{|w|}{2}$. If the sequence p_n was really converging to (q, w), then there would be some $N \in \mathbb{N}$ such that for all $n \geq N$, $d((x_n, 0), (q, w)) < \epsilon$. However, this cannot happen since the points of the form $(x_n, 0)$ are on the x-axis but all points in $M_{\frac{w}{2}}((q, w))$ are strictly above the x-axis. This is a contradiction, and therefore we must have the limit of p_n lying on the x-axis. ■

The set is not open because if you pick any point $q \in \mathbb{R}$, any "ball" around p does not contain only points of $\mathbb{R}$ – there is always a $q \in \mathbb{R}^2$ such that q is in the ball but not in $\mathbb{R}$:



2. Let (M, d) be a metric space and let $S = \emptyset$ (i.e. the empty set). Is $\emptyset$ an open set, a closed set, both open and closed, or neither open nor closed? Why or why not?
   Solution: $\emptyset$ is both open and closed. It is closed because for all convergent sequences of point in $\emptyset$ (there are none...), the limit (which doesn't exist since no sequence exist) is also in $\emptyset$. It is open because given any point in $\emptyset$ (there isn't one), there is a ball small enough so that all points in the ball are also points of $\emptyset$ (but any ball is precisely $\emptyset$ here and $\emptyset \subset \emptyset$).

3. Let (M, d) be a metric space and let $x \in M$ be some arbitrary point. Prove that $S = \{x\}$ is a closed set.

Solution: Let p_n be a sequence in S. Since S contains only one point, we know this sequence *must* be $p_1 = x, p_2 = x, p_3 = x, \ldots$. This sequence converges to x: let $\epsilon > 0$ and choose $N = 1$, then compute

$$d(p_n, p) = d(p_n, p) = d(x, x) = 0 < \epsilon.$$

But since (p_n) is the only sequence in S and its limit lies in S, it follows that the limit (i.e. x) of all convergent sequences (there's only one) in S lies in S. This means S contains all of its limits. This means that S is closed.