Homework 3 — MATH 4590 Spring 2018

1. Prove that if $M = \mathbb{R}$ and $d(x, y) = |x - y|$, then (M, d) is a metric space.
   Solution: We must show three things:

   a.) (Positive definite) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$

   b.) (Symmetric) $d(x, y) = d(y, x)$. and

   c.) (Triangle inequality) $d(x, y) \leq d(x, z) + d(z, y)$

   To prove *a.*), note that the absolute value of any real number is always ≥ 0. Furthermore, if $|x - y| = 0$, then either $x - y = 0$ or $-(x - y) = 0$; in either case we have $x = y$. On the other hand, if $x = y$, then it is clear that $|x - y| = |0| = 0$. This proves *a.*).
   To prove *b.*), it is sufficient to note that the absolute value is symmetric, i.e. for any $x, y \in \mathbb{R}$, $|x - y| = |y - x|$.

   To prove *c.*), follow the proof we did in class (or in the text). ■

2. Prove the following "ϵ-principle": if for all $\epsilon > 0$ it holds that $|x - y| < \epsilon$, then $x = y$.
   Proof: Suppose that $x \neq y$. Then $|x - y| > 0$. Choose ϵ so that $0 < \epsilon < |x - y|$ (for instance, $\epsilon = \dfrac{|x-y|}{2}$ would work). But since for all ϵ, $|x - y| \leq \epsilon$ we get
   $$0 < \epsilon < |x - y| \leq \epsilon.$$
   From this we must conclude $\epsilon < \epsilon$, which is absurd. Hence we have a contradiction. Therefore it must be the case that $x = y$. ■

3. Prove that if $M = \mathbb{R}^2$ and $d((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$, then (M, d) is a metric space. Draw the "unit circle" in this metric space, i.e. the set $\{x \in \mathbb{R}^2 : d(x, 0) = 1\}$.
   Proof: To show *a.*)), note that just as before, the absolute value function is non-negative by definition. Furthermore, if $d(x, y) = 0$, then both $|x_1 - x_2| = 0$ and $|y_1 - y_2| = 0$, from which we may conclude that $x_1 = x_2$ and $y_1 = y_2$, in other words, $(x_1, y_1) = (x_2, y_2)$. Conversely, assume that $(x_1, y_1) = (x_2, y_2)$. Then
   $$d((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\} = \max\{0, 0\} = 0,$$
   completing the proof of *a.*).

   To prove *b.*), simply note that
   $$d((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\} = \max\{|x_2 - x_2|, |y_2 - y_1|\} = d((x_2, y_2), (x_1, y_1)).$$
   To prove *c.*), we apply the triangle inequality for the absolute value in each piece of the maximum function:
   $$|x_1 - x_2| \leq |x_1 - x_3| + |x_3 - x_2|,$$

and

$$|y_1 - y_2| \le |y_1 - y_3| + |y_3 - y_2|.$$

Thus,

$$\begin{aligned} d((x_1, y_1), (x_2, y_2)) &= \max\{|x_1 - x_2|, |y_1 - y_2|\} \\ &\le \max\{|x_1 - x_3| + |x_3 - x_2|, |y_1 - y_3| + |y_3 - y_2|\} \\ &\le \max\{|x_1 - x_3|, |y_1 - y_3|\} + \max\{|x_3 - x_2|, |y_3 - y_2|\} \\ &= d((x_1, y_1), (x_3, y_3)) + d((x_2, y_2), (x_3, y_3)), \end{aligned}$$

completing the proof. ■

4. Let (M, d) be any metric space and let (x_n) and (y_n) be convergent sequences in M with $x_n \to x$ and $y_n \to y$. Prove that the sequence $(d(x_n, y_n))$ is a convergent sequence in $\mathbb{R}$ with $d(x_n, y_n) \to d(x, y)$.
(*note: you don't have a formula for the metric d in M nor do you know much about M – it could be a lot of things – but you should be thinking of the metric in $\mathbb{R}$ as $d_{\mathbb{R}}(x, y) = |x - y|$*)
Proof: Recall the reverse triangle inequality:

$$|d(x, z) - d(y, z)| \le d(x, y).$$

Since $x_n \to x$, we know there exists an N_x such that for all $n \ge N_x$, $d(x_n, x) < \dfrac{\epsilon}{2}$. Similarly, since $y_n \to y$, there is an N_y such that for all $n \ge N_y$, $d(y_n, y) < \dfrac{\epsilon}{2}$.

Now we shall prove that $d(x_n, y_n) \to d(x, y)$. To do this, let $\epsilon > 0$ and choose $N \ge \max\{N_x, N_y\}$ and let $n \ge N$. It remains to show that $|d(x_n, y_n) - d(x, y)| < \epsilon$. To see this, compute

$$\begin{aligned} |d(x_n, y_n) - d(x, y)| &= |d(x_n, y_n) - d(x, y) + d(x_n, y) - d(x_n, y)| \\ &\le |d(x_n, y_n) - d(x_n, y)| + |d(x_n, y) - d(x, y)| \\ &\overset{\text{reverse } \Delta \text{ ineq}}{\le} d(y_n, y) + d(x_n, x) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon, \end{aligned}$$

completing the proof. ■

5. Consider the metric space $(\mathbb{R}, d)$ where $d(x, y) = |x - y|$. Prove that the function $f \colon \mathbb{R} \to \mathbb{R}$ with $f(x) = x$ is continuous.
Proof: Let $\epsilon > 0$ and let $p \in \mathbb{R}$. Choose $\delta = \epsilon$. Then compute for $q \in \mathbb{R}$ with $|q - p| < \delta$:

$$|f(p) - f(q)| = |p - q| < \delta = \epsilon,$$

completing the proof. ■