Homework 11 — MATH 4590 Spring 2018

1. In this problem we will sketch the graphs of $Q_n(x)$ (from 16 April slides).

   (a) First compute $\int_{-1}^{1}(1-x^2)^n \mathrm{d}x$ for $n = 0, 1, 2, 3, 4, 5$ (Using WolframAlpha for this is OK). This defines the constants $c_n = \dfrac{1}{\int_{-1}^{1}(1-x^2)^n \mathrm{d}x}$.

   (b) Write down $Q_n(x) = c_n(1-x^2)^n$ for $n = 0, 1, 2, 3, 4, 5$.

   (c) Draw $Q_n(x)$ (from 16 April slides) on $[-1, 1]$ for $n = 0, 1, 2, 3, 4, 5$ (ok to use WolframAlpha or Desmos for this).

2. In this problem you will argue that $(1-x^2)^n \geq 1 - nx^2$ as needed in the proof of the Weierstrass approximation theorem (16 April slides). To do it, consider the function $g\colon [0,1] \to \mathbb{R}$ defined by

$$(*) \qquad g(x) = (1-x^2)^n - 1 + nx^2.$$

   (a) Differentiate this function.

   (b) Is the derivative positive or negative?

   (c) What does that mean (in a calc 1 sense)?

   (d) What is the value of $g(0)$?

   (e) Combining your answers to part (c) and part (d), what must you conclude about $g(x)$ for $x \in [0, 1]$?

   (f) What do you do with your answer to (e) to arrive at the desired inequality $(*)$?

3. It is said in the 16 April slides that for any $0 < \delta < 1$, the function $Q_n\colon [0,1] \to \mathbb{R}$ defined by $Q_n(x) = c_n(1-x^2)^n$ obeys $Q_n \rightrightarrows 0$ on $[\delta, 1]$. In this problem, we investigate that further.

   (a) It was shown in the slides that $c_n < \sqrt{n}$. Therefore

$$Q_n(x) \leq \sqrt{n}(1-x^2)^n.$$

   Use L'Hôpital's rule to argue that for any $x \in [\delta, 1]$,

$$\lim_{n\to\infty} Q_n(x) = 0.$$

   (*hint*: rewrite the function as $\dfrac{\sqrt{n}}{(1-x^2)^{-n}}$ before trying L'Hôpital's rule. Also be careful when differentiating... it is **not** a "power rule" since you are differentiating w.r.t n.)

(b) Does $Q_n \rightrightarrows 0$ on $[0,1]$? Why or why not? (A graphical argument about "ϵ-tubes" combined with the graphs in question 1 is ok)

(c) Why do we know that $Q_n \rightrightarrows 0$ on $[\delta,1]$? (again, a graphical argument is ok)

4. Recall the definition of Bernstein polynomials from the 16 April slides. Let $f(x) = \cos(2\pi x)$. Find and plot the first 5 Bernstein polynomials associated with $\cos(2\pi x)$ on the interval $[0,1]$.