(i) Pointwise limit:

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x^n}{1+x^n} = \begin{cases} 0; & 0 \le x < 1 \\ \frac{1}{2}; & x = 1 \\ 1; & 1 < x \le 2 \end{cases}$$

Because if $0 \le x < 1$, then $\left\{ \underset{n\to\infty}{x^n \xrightarrow{\text{as}} 0} \right\}$.

If $x=1$, then $\frac{x^n}{1+x^n} = \frac{1^n}{1+1^n} = \left\{ \frac{1}{2} \xrightarrow[n\to\infty]{\text{as}} \frac{1}{2} \right\}$

if $1 < x \le 2$, then $\left\{ x^n \xrightarrow[n\to\infty]{\text{as}} \infty \right\}$ but this means that

$\frac{x^n}{1+x^n} \to \left\{ \frac{\infty}{1+\infty} = \frac{\infty}{\infty} \text{ as } n\to\infty \right\}$ So, by L'Hôpital's rule,

$$\lim_{n\to\infty} \frac{x^n}{1+x^n} \overset{\text{L.H.}}{=} \lim_{n\to\infty} \frac{nx^{n-1}}{0+nx^{n-1}} = \lim_{n\to\infty} 1 = 1$$

Why convergence is not uniform:

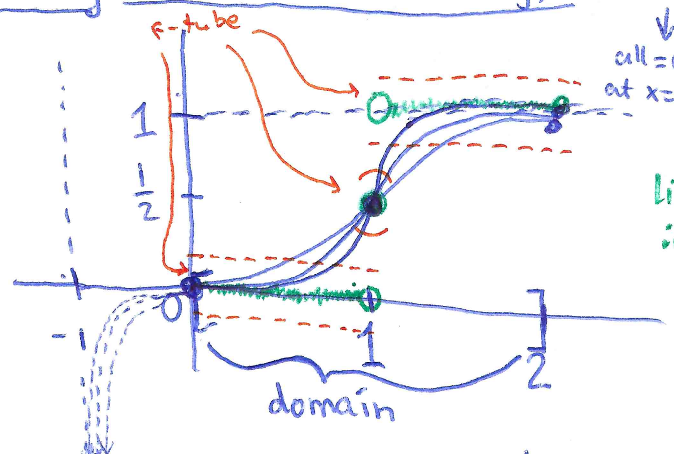
<u>One way to explain (graphically):</u> $\frac{x^n}{1+x^n}$

all $= \frac{1}{2}$ at $x=1$

asymptote at $x=-1$ when n odd

no asymptote when n even

all $=0$ at $x=0$



Limit function in green

The function never enters the ε-tube!

Another way to explain (via theorem): we know (Thm:) if $f_n : A \to \mathbb{R}$ is a sequence of ~~fxxx~~ continuous functions ~~then~~ and $f_n \rightrightarrows f$, then f is continuous.

In this problem, each f_n is continuous (on its domain) but f is not continuous. Since f is not continuous, ~~the~~ we must have $f_n \to f$ (we calculated that, so it's ok) while $f_n \not\rightrightarrows f$.

(2) Recall the characteristic function $\chi_E(t) = \begin{cases} 1, & t \in E \\ 0, & t \notin E \end{cases}$

The functions $f_n : [0,1] \to \{0,1\}$ defined by $f_n(t) = \dfrac{\chi_{\mathbb{Q}\cap[0,1]}(t)}{n}$ is discontinuous everywhere on $[0,1]$ for all $n = 1, 2, \ldots$,
But, its limit function is zero identically, i.e. $f_n \to 0$.
In fact, this convergence is uniform, i.e. $f_n \rightrightarrows 0$.

(3) $\begin{Bmatrix} g_n : [0,\infty) \to \mathbb{R} \\ g_n(x) = \frac{1}{n} e^{-nx} \end{Bmatrix}$

$$\boxed{\begin{array}{l} \text{Limit function: } g(x) = \lim_{n\to\infty} g_n(x) = \cancel{\frac{e^{-nx}}{n}} = \lim_{n\to\infty} \frac{e^{-nx}}{n} = 0 \\ \text{Derivative of limit fnct: } g'(x) = 0 \end{array}}$$

$$g_n'(x) = \frac{d}{dx} \frac{e^{-nx}}{n} = \frac{ne^{-nx}}{n} = e^{-nx}$$

and

$$\lim_{n\to\infty} g_n'(x) = \lim_{n\to\infty} e^{-nx} = 0$$

In other words:

$$\begin{tikzcd} g_n(x) \arrow[r, "\frac{d}{dx}"] \arrow[d, "\lim_{n\to\infty}"'] & e^{-nx} \arrow[d, "\lim_{n\to\infty}"] \\ 0 \arrow[r, "\frac{d}{dx}"'] & 0 \end{tikzcd}$$

(4) Let $\left\{\begin{array}{l} g_n : [0,1] \to \mathbb{R} \\ g_n(x) = nx(1-x)^n \end{array}\right\}$

$x = 1-u$

$u = 1-x$ $\qquad x=0 \to u=1$

$-du = dx$ $\qquad x=1 \to u=0$

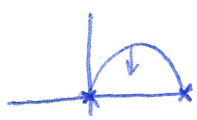
at $x=0 \to g_n(x)=0$

So $g_n(1)=0$

$g_n(\tfrac{1}{2}) = \frac{n}{2}\cdot\left(\tfrac{1}{2}\right)^n \to 0$ as $n\to\infty$

for any $x\in(0,1)$, $|1-x|<1$,

so $\lim_{n\to\infty}(1-x)^n = 0$



$$\int_0^1 g_n(x) = \int_0^1 nx(1-x)^n\,dx = -\int_1^0 n(1-u)u^n\,du$$

$$= n\int_0^1 u^n - u^{n+1}\,du$$

$$= n\left[\frac{u^{n+1}}{n+1} - \frac{u^{n+2}}{n+2}\right]_0^1$$

$$= n\left[\left(\frac{1}{n+1} - \frac{1}{n+2}\right) - 0\right]$$

$$= n\left[\frac{(\cancel{n}+2)-(\cancel{n}+1)}{(n+1)(n+2)}\right]$$

$$= \frac{n}{(n+1)(n+2)} = \frac{n}{n^2+3n+2}$$

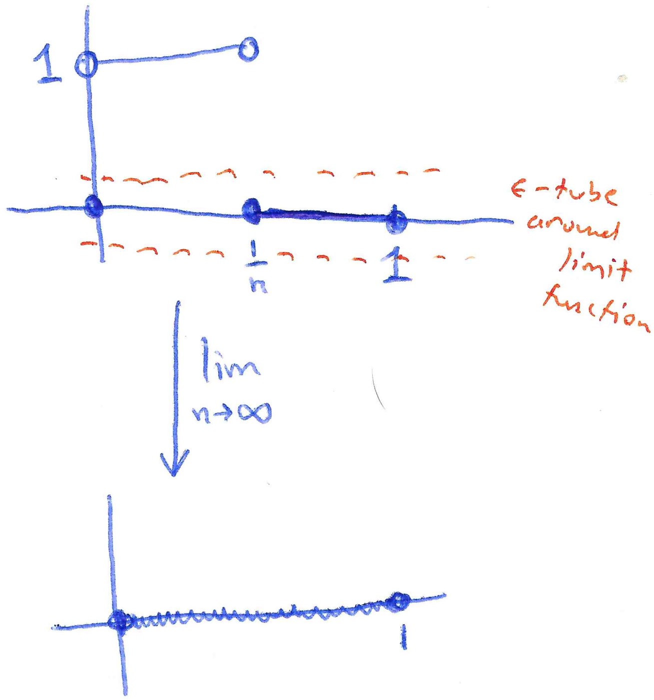
<u>Limit fnct:</u> $\lim_{n\to\infty} g_n(x) = 0$ (for all $x\in[0,1]$)

$\int_0^1 0\,dx = 0 \leftarrow$ integral of limit fnct

$\lim_{n\to\infty}\int_0^1 g_n(x)\,dx = 0$, so ....

$$\begin{tikzcd} g_n(x) \arrow[r, "\int_0^1"] \arrow[d, "\lim_{n\to\infty}"'] & \frac{n}{n^2+3n+2} \arrow[d, "n\to\infty"] \\ 0 \arrow[r, "\int_0^1"'] & 0 \end{tikzcd}$$

(5)



not uniform because the sequence never enters ~~the~~ any sufficiently small ε-tube