1. Chapter 1, # 1: Prove that for all sets A, B, C, the formula

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

is true.
Proof: It suffices to show that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ and $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$. Assume that $x \in A \cup (B \cap C)$, thus either $x \in A$ or $x \in B \cap C$. If $x \in A$, then $x \in A \cup B$ and $x \in A \cup C$, therefore $x \in (A \cup B) \cap (A \cup C)$. Now suppose that $x \in B \cap C$. Then $x \in B$ and $x \in C$, and so both $x \in A \cup B$ and $x \in A \cup C$, hence $x \in (A \cup B) \cap (A \cup C)$. Therefore $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

Now we show that $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$. Assume that $x \in (A \cup B) \cap (A \cup C)$. Then $x \in A \cup B$ and $x \in A \cup C$. If $x \in A$, then $x \in A \cup (B \cap C)$, so assume that $x \notin A$.

Claim: $x \in B \cap C$
Proof of claim: Suppose that $x \notin B \cap C$. Then $x \notin B$ and $x \notin C$. Since $x \notin A$, it follows that both $x \notin A \cup B$ and $x \notin A \cup C$, a contradiction. Therefore $x \in B \cap C$.

Since $x \in B \cap C$, it follows that $x \in A \cup (B \cap C)$. Therefore $A(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$. Therefore we have shown that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, completing the proof. ■

2. Chapter 1, # 2:

(a) Prove that $(A^c)^c = A$.
Proof: Let A^c denote the complement of A in X, i.e. $A^c = X \setminus A$. We shall show that $(A^c)^c \subseteq A$ and $A \subseteq (A^c)^c$. Let $x \in (A^c)^c$. Then $x \in X \setminus A^c$. But $X \setminus A^c = X \setminus (X \setminus A)$.

Claim: $X \setminus (X \setminus A) = A$
Proof of claim: If $z \in X \setminus (X \setminus A)$, then z is in the set of elements that are not in $X \setminus A$. Those elements are exactly the ones in A, by definition. Conversely, if $z \in A$, then $z \notin X \setminus A$, and so z is not removed when forming $X \setminus (X \setminus A)$, showing that $A \subset X \setminus (X \setminus A)$. Thus $X \setminus (X \setminus A) = A$.

Since $x \in X \setminus (X \setminus A)$, we see from the claim that $x \in A$. Thus we have shown that $(A^c)^c \subseteq A$.

Now let $x \in A$. Then $x \notin X \setminus A$. Therefore $x \in X \setminus (X \setminus A)$. Since $X \setminus (X \setminus A) = X \setminus A^c = (A^c)^c$, we see that $A \subseteq (A^c)^c$. Therefore $(A^c)^c = A$, completing the proof. ■

(b) Prove **deMorgan's law**: Prove

$$(A \cap B)^c = A^c \cup B^c \tag{1}$$

and derive from it the law

$$(A \cup B)^c = A^c \cap B^c. \tag{2}$$

Solution: First let us prove (1). Let $x \in (A \cap B)^c$. Then $x \in X \setminus (A \cap B)$. This means that $x \notin A \cap B$. Without loss of generality, assume that $x \notin A$. Then $x \in A^c$, and so $x \in A^c \cup B^c$.

Now let $x \in A^c \cup B^c$. Then $x \in A^c$ or $x \in B^c$. Without loss of generality, asusme that $x \in A^c$. Then $x \notin A$, so $x \notin A \cap B$. Thus $x \in (A \cap B)^c$. Completing the proof. ■

To derive (2) from (1), replace A with A^c and B with B^c in (1) to get

$$(A^c \cap B^c)^c = (A^c)^c \cup (B^c)^c. \tag{3}$$

On the right-hand-side note that by part (a), we know $(A^c)^c = A$ and $(B^c)^c = B$, so we get

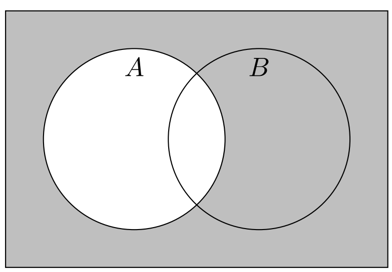
$$(A^c \cap B^c)^c = A \cup B.$$

Taking the complement of each side and applying part (a) to the left-hand side yields the desired result:

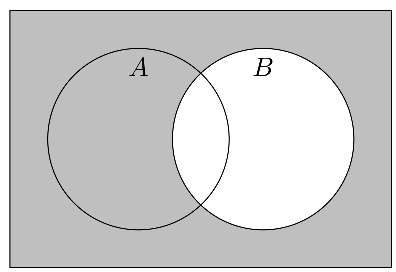
$$A^c \cap B^c = (A \cup B)^c.$$

(c) Draw Venn diagrams to illustrate the two laws.
Solution:

A^c:

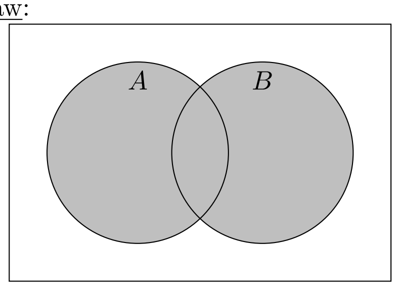


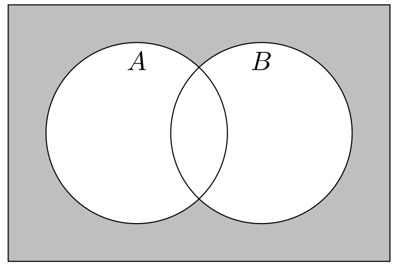
B^c:



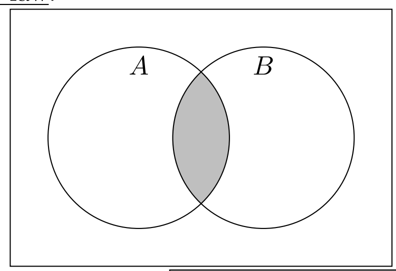
First law:

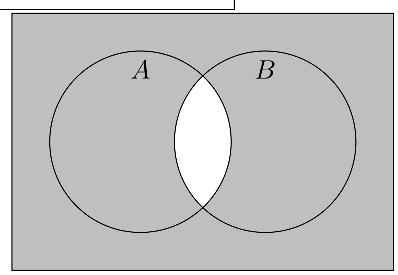
$A \cup B$:



$(A \cup B)^c = A^c \cap B^c$: 

Second law:

$A \cap B$: 

$(A \cap B)^c = A^c \cup B^c$: 

(d) Generalize these laws to more than two sets.
Solution: Suppose we have three sets A, B, C and we want to look at $(A \cap B \cap C)^c$. Write $X = B \cap C$, and observe

$$(A \cap B \cap C)^c = (A \cap X)^c \overset{(1)}{=} A^c \cap X^c = A^c \cap (B \cap C)^c \overset{(1)}{=} A^c \cap B^c \cap C^c.$$

This leads us to a conjecture:
Conjecture: The following formula holds for $n = 1, 2, 3, \ldots$:

$$(A_1 \cap A_2 \cap \ldots \cap A_n)^c = A_1^c \cup A_2^c \cup \ldots \cup A_n^c.$$

Proof: The case $n = 1$ holds trivially. The case $n = 2$ was proven in part (b). The case $n = 3$ was proven in the beginning of part (d). Assume the formula holds for $n = N$. We now prove it holds for $n = N + 1$:

Claim: If for all sets, $A_1, \ldots, A_N$,

$$(A_1 \cap \ldots \cap A_N)^c = A_1^c \cup \ldots \cup A_N^c,$$

then for any additional set A_{N+1},

$$(A_1 \cap \ldots \cap A_N \cap A_{N+1})^c = A_1^c \cup \ldots \cup A_N^c \cup A_{N+1}^c.$$

Proof of claim: Define $\tilde{A_N} = A_N \cap A_{N+1}$. Then we see using the hypothesis of this claim that

$$(A_1 \cap \ldots \cap \tilde{A_N})^c = A_1^c \cup \ldots \cup \tilde{A_N}^c = A_1^c \cup \ldots \cup (A_N \cup A_{N+1})^c,$$

and so from the $n = 2$ case of the conjecture (proved earlier), we may conclude

$$(A_1 \cap \ldots \cap \tilde{A_N})^c = A_1^c \cup \ldots \cup A_N^c \cup A_{N+1}^c,$$

completing the proof of the claim.

Since this claim holds and the base cases $n = 1$, $n = 2$, and $n = 3$ hold, we have shown via induction that the conjecture holds. ■

3. Chapter 1, # 6: Why is the square of and odd integer odd and the square of an even integer even? What is the situation for higher powers?

   Solution: Recall that any positive natural number has a unique prime factorization $n = p_1^{e_1} p_2^{e_2} \ldots p_i^{e_i}$, where the $p_1, \ldots, p_i$ are prime numbers with exponents $e_1, \ldots, e_i \in \mathbb{N}$. Since 2 is prime and being even means being divisible by 2, a number is even if and only if one of its prime factors $p_1, \ldots, p_i$ is equal to 2.

   If n is an odd integer, then when writing $n = p_1^{e_1} p_2^{e_2} \ldots p_i^{e^i}$, we observe that $p_1 \neq 2$ and $p_2 \neq 2$ and ... and $p_i \neq 2$ (otherwise n would be even). Now consider the square of n:

$$n^2 = (p_1^{e_1} p_2^{e_2} \ldots p_i^{e_i})^2 = (p_1^{e_1})^2 (p_2^{e_2})^2 \ldots (p_i^{e_i})^2 = p_1^{2e_1} p_2^{2e_2} \ldots p_i^{2e_i}.$$

   We see that the only prime factors of n^2 are the same ones that are factors of n. Since 2 was not a factor of n, it follows that 2 is not a factor of n^2.

   A similar argument shows that the square of an even must be even. This same argument holds for higher powers because

$$n^\ell = (p_1^{e_1} p_2^{e_2} \ldots p_i^{e_i})^\ell = p_1^{\ell e_1} p_2^{\ell e_2} \ldots p_i^{\ell e_i}.$$

4. In this problem, there is a known subset of $\mathbb{R}$ called $\mathscr{C}$ which has an upper bound. The set C is defined by

$$C = \{a \in \mathbb{Q}\colon \text{for some cut } A|B \in \mathscr{C}, a \in A\}$$

   and the set D is defined by

$$D = \mathbb{Q} \setminus C.$$

   (a) Claim 1: $C|D$ is a cut.

   Solution: We must argue that

   a.) $C \cup D = \mathbb{Q}$, $C \neq \emptyset$, $D \neq \emptyset$, and $C \cap D = \emptyset$, b.) if $c \in C$ and $d \in D$, then $c < d$, and

c.) C contains no largest element.

By definition, $C \subset \mathbb{Q}$ and $C \cap D = \emptyset$. To prove a.), it follows from the definition of D that $C \cup D = \mathbb{Q}$. We know that $C \neq \emptyset$ and $D \neq \emptyset$ because C is bounded above.

To prove b.), suppose there is a $c \in C$ and a $d \in D$ with $d \leq c$. Let $c = C_1|C_2$ and $d = D_1|D_2$. Then we may conclude that $D_1 \subseteq C_1$. But we cannot have $D_1 = C_1$, because it would follow that $D_2 = C_2$ and hence $c = d$ which contradicts the fact that $C \cap D = \emptyset$. If $D_1 \subsetneq C_1$, then by the definition of C, $d \in C$. But this cannot happen because $D = \mathbb{Q} \setminus C$. Hence we have shown that $D_1 \not\subset C_1$ and thus it is not true that $d \leq c$. Therefore b.) holds.

To prove c.), suppose that C does contain a largest element – call it $\tilde{c}$. Since $\tilde{c} \in \mathbb{Q}$ we can express it as $\dfrac{p}{q}$ for $p, q \in \mathbb{Z}$ with $q \neq 0$. By the definition of C, there is some cut $A|B \in \mathscr{C}$ for which $\dfrac{p}{q} \in A$.

> Claim: $A|B = \tilde{c}$
>
> *Proof*: Since $\dfrac{p}{q} \in A$, it follows that $\tilde{c} \leq A|B$. Now we show that $A|B \leq \tilde{c}$. Suppose that $A|B > \tilde{c}$. Then if we write $\tilde{c} = \tilde{C}|\tilde{D}$, it follows that $\tilde{C} \subsetneq A$. But this means there is some larger $\hat{c} = \hat{C}|\hat{D} \in A$ with $\tilde{C} \subsetneq \hat{C} \subseteq A$. But this means that $\hat{c} \in C$ is a larger element of C than $\tilde{c}$, a contradiction to $\tilde{c}$ being the largest. Therefore $A|B \leq \tilde{c}$ and since we proved earlier that $\tilde{c} \leq A|B$, we may conclude that $A|B = \tilde{c}$.

So we see that may write the real number $\tilde{c}$ as the cut

$$\tilde{c} = A|B = \left(-\infty, \frac{p}{q}\right) \cap \mathbb{Q} \,\Bigg|\, \left[\frac{p}{q}, \infty\right) \cap \mathbb{Q}.$$

From this we observe that $\tilde{c} \notin A$, a contradiction. Therefore $\tilde{c}$ cannot exist, i.e. C has no largest element.

Therefore a.), b.), and c.) hold and we may conclude that $C|D$ is a cut. ■

5. Claim 2: $C|D$ is an upper bound for $\mathscr{C}$
   Proof: Let $E|F \in \mathscr{C}$. We will show that $E|F \leq C|D$. Since $E|F \in \mathscr{C}$, we know by the definition of C that for all $a \in E$, $a \in C$, hence $E \subseteq C$. Therefore $E|F \leq C|D$. ■