MATH 4590 - EXAM 2 - SPRING 2018
SOLUTION

23 March 2018
Instructor: Tom Cuchta

Instructions:

- Show all work, clearly and in order, if you want to get full credit. If you claim something is true **you must show work backing up your claim**. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers algebraically whenever possible to ensure full credit.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point.
- Good luck!

1. (15 points) Write the definition.

   (a) (3 points) Let (M, d) be a metric space and let $S \subset M$. Define what a limit of S is.
   Solution: We say that $p \in M$ is a limit of S provided there is a sequence (p_n) in S such that $p_n \to p$.

   (b) (3 points) Let (M, d) be a metric space and let $S \subset M$. Define what it means for S to be a closed set.
   Solution: We say that S is closed if it contains all of its limits.

   (c) (3 points) Let (M, d) be a metric space and let $S \subset M$. Define what it means for S to be an open set.
   Solution: We say that S is open if $\forall p \in S \exists r > 0$ so that $\{q \in M : d(p, q) < r\} \subset S$.

   (d) (3 points) Define what $\lim_{t \to a} f(t) = L$ means.
   Solution: $\forall \epsilon > 0 \exists \delta > 0$ such that$|t - a| < \delta$ implies $|f(t) - L| < \epsilon$.

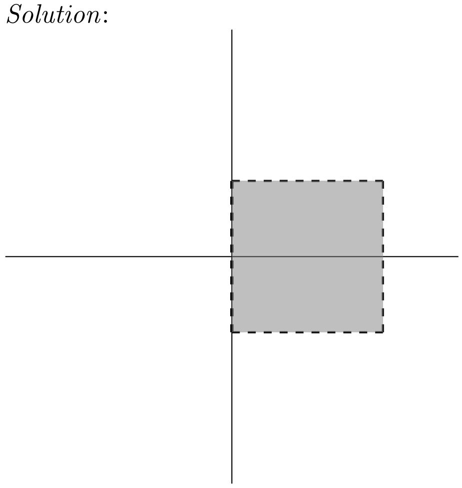
   (e) (3 points) Let $f : \mathbb{R} \to \mathbb{R}$. Define the derivative of f at x.
   Solution: $f'(x) = \lim_{t \to x} \dfrac{f(t) - f(x)}{t - x}$

2. (18 points) For each, sketch a picture of S and decide whether S is open, closed, both open and closed, or neither open nor closed.

   (a) (6 points) Consider the metric space $(M, d) = (\mathbb{R}^2, d)$, where $d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ is the (usual) Eulcidean metric. Let

$$S = \{(x, y) \in \mathbb{R}^2 : 0 < x < 2, -1 < y < 1\}.$$

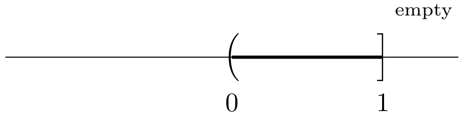
   Solution:



   Open – for any $p \in S$ choose r to be the shortest distance to the boundary and divide it by 2.
   Not closed – the point $(1, 0) \notin S$ while the sequence $(0, 1 - \frac{1}{n})$ is in S and its limit is $(1, 0)$.

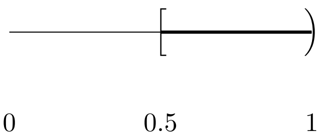
   (b) (6 points) Consider the metric space $(M, d) = (\mathbb{R}, d)$, where $d(x, y) = |x - y|$ is the usual metric on the real line. Let $S = \bigcup_{n=1}^{\infty} \left(\frac{1}{n}, 1\right]$.

   Solution: Notice that $S = \bigcup_{n=1}^{\infty} = \underbrace{\Big(1, 1\Big]}_{\text{empty}} \cup \left(\frac{1}{2}, 1\right] \cup \left(\frac{1}{3}, 1\right] \cup \ldots = \Big(0, 1\Big]$



   Not open – if $p = 1$ then for *any* radius r, the set $\{q \in \mathbb{R} : |1 - q| < r\}$ always contains points to the right of 1, so it is never a subset of S.
   Not closed – the point 0 is a limit of S because it is a limit of the sequence $(\frac{1}{n})$ but $0 \notin S$.

(c) (6 points) Consider the metric space $(M, d) = ((0,1), d)$, where $d(x,y) = |x-y|$ is the usual metric on the real line. Let $S = [0.5, 1)$.
Solution:



Not Open – on the left this is "obvious"; taking $p = 0.5$, the ball determined by radius chosen will include points to the left of 0.5.
Closed – not as obvious here. We cannot say that 1 is a limit of S because 1 is not in S! Therefore, S contains all of its limits. The sequence $p_n = 1 - \frac{1}{n}$ here does not converge: there is nothing to converge to. Notice in the definition of p being a limit of S requires "$p \in M$".

3. (12 points) Prove that $\lim_{x \to 3} 2x^2 + x - 1 = 20$.
Solution: Let $\epsilon > 0$. Choose $0 < \delta < \min\left\{\frac{\epsilon}{15}, 1\right\}$. Compute

$$\begin{aligned}
|(2x^2 + x - 1) - 20| &= |(2x^2 - 18) + (x-3)| \\
&= |2(x^2 - 9) + (x-3)| \\
&= |2(x-3)(x+3) + (x-3)| \\
&= |x-3||2(x+3) + 1| \\
&= |x-3||2(x+3-3+3) + 1| \\
&= |x-3||2(x-3+6) + 1| \\
&= |x-3||2(x-3) + 13| \\
&\leq |x-3|(2|x-3| + 13) \\
&< \delta(2\delta + 13) \\
&< 15\delta \\
&< \epsilon.
\end{aligned}$$

4. (12 points) Prove that the intersection of finitely many open sets is open. (*note: you may use the result from HW5 that says that the union of finitely many closed sets is closed*).
Proof: Let $\mathcal{O}_1, \ldots, \mathcal{O}_n$ be a finite list of open sets. Then $\mathcal{O}_1^c, \ldots, \mathcal{O}_n^c$ is a finite list of closed sets (since the complement of an open set is a closed set). By the result from HW5, we know that $\mathcal{O}_1^c \cup \ldots \cup \mathcal{O}_n^c$ is a closed set. Therefore $(\mathcal{O}_1^c \cup \ldots \cup \mathcal{O}_n^c)^c$ is an open set. But,

$$(\mathcal{O}_1^c \cup \ldots \cup \mathcal{O}_n^c)^c = (\mathcal{O}_1^c)^c \cap \ldots \cap (\mathcal{O}_n^c)^c = \mathcal{O}_1 \cap \ldots \cap \mathcal{O}_n$$

is an open set, completing the proof. ■

5. (12 points) Use induction to prove that

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} x e^{-x} = (-1)^n e^{-x}(x - n).$$

(*note: you may use the basic rules from earlier calculus courses here – no need to use the definition of the derivative*)
Proof: Let $n = 0$, then the left-hand side is

$$\underbrace{\frac{\mathrm{d}^0}{\mathrm{d}x^0} x e^{-x}}_{\text{"don't differentiate"}} = x e^{-x},$$

while the right-hand side is $(-1)^0 e^{-x}(x - 0) = x e^{-x}$. Therefore the $n = 0$ case holds. Now assume that the formula holds for $n = N$. Calculate

$$\begin{aligned}
\frac{\mathrm{d}^{N+1}}{\mathrm{d}x^{N+1}} &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}^N}{\mathrm{d}x^N} xe^{-x}\right] \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left[(-1)^N e^{-x}(x-N)\right] \\
&= (-1)^N \frac{\mathrm{d}}{\mathrm{d}x}\left[e^{-x}(x-N)\right] \\
&\overset{\text{prod. rule}}{=} (-1)^N \left[-e^{-x}(x-N) + e^{-x}\right] \\
&= (-1)^N e^{-x}[N - x + 1] \\
&= (-1)^{N+1} e^{-x}(x - (N+1)),
\end{aligned}$$

completing the proof. ■

6. (20 points) In this problem we investigate the Taylor approximation to the cosine function.
   (a) (5 points) Find the 4th order Taylor polynomial of $f(x) = \cos(x)$.
   Solution: Calculate derivatives: $f'(x) = -\sin(x)$, $f''(x) = -\cos(x)$, $f'''(x) = \sin(x)$, $f^{(4)}(x) = \cos(x)$. Therefore the 4th order Taylor polynomial is

$$\begin{aligned}
\sum_{k=0}^{4} \frac{f^{(k)}(x)}{k!} h^k &= \frac{f^{(0)}(x)}{0!} h^0 + \frac{f^{(1)}(x)}{1!} h + \frac{f^{(2)}(x)}{2!} h^2 + \frac{f^{(3)}(x)}{3!} h^3 + \frac{f^{(4)}(x)}{4!} h^4 \\
&= \cos(x) - \sin(x)h - \frac{\cos(x)h^2}{2!} + \frac{\sin(x)h^3}{3!} + \frac{\cos(x)h^4}{4!}.
\end{aligned}$$

   (b) (5 points) Set $x = 0$ in the formula (this "centers" it at zero).
   Solution: Setting $x = 0$ yields

$$1 - 0 - \frac{h^2}{2!} + 0 + \frac{h^4}{4!} = 1 - \frac{h^2}{2!} + \frac{h^4}{4!}$$

   (c) (5 points) Use the Taylor approximation theorem to write a formula (in terms of θ) for the remainder R at $\frac{1}{2}$.
   Solution: $R\left(\frac{1}{2}\right) = \frac{f^{(5)}(\theta)}{5!}\left(\frac{1}{2}\right)^5 = \frac{-\sin(\theta)}{5!}\left(\frac{1}{2}\right)^5$

   (d) (5 points) Find a number $\xi \in \mathbb{R}$ such that $\left|R\left(\frac{1}{2}\right)\right| \leq \xi$.
   Solution:

$$\left|R\left(\frac{1}{2}\right)\right| = \left|\frac{-\sin(\theta)}{5!}\left(\frac{1}{2}\right)^5\right| \leq \frac{1}{2^5 5!}.$$

Pick ONE of these two problems to do – **cross out** the one you don't want me to grade.

7. (11 points) Prove that if F is closed and K is compact, then $F \cap K$ is compact.
Proof: Let (a_n) be a sequence in $F \cap K$. Since (a_n) is in K, it has a convergent subsequence (a_{n_k}) in K. But (a_{n_k}) is a subsequence of a sequence in $F \cap K$, so it also lies in $F \cap K$. Since (a_{n_k}) is a convergent sequence in F, it follows that its limit is also in F and hence also in $F \cap K$. ■

8. (11 points) Prove that $|\sin(x) - \sin(y)| \leq |x - y|$ by applying the mean value theorem to the function $f(x) = \sin(x)$ on the interval $[x, y]$.
Proof: By the mean value theorem, there is a $\theta \in [x, y]$ so that

$$\frac{\sin(x) - \sin(y)}{x - y} = \sin'(\theta) = \cos(\theta).$$

Taking absolute value and multiplying by $|x - y|$ yields

$$|\sin(x) - \sin(y)| = \underbrace{|\cos(\theta)|}_{\leq 1} |x - y| \leq |x - y|. \blacksquare$$