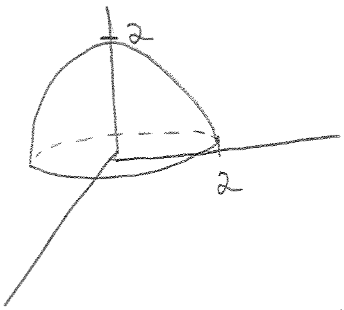
Section 6.6 # 282

$S \to$ hemisphere $\begin{cases} x^2+y^2+z^2=4 \\ z \geq 0 \end{cases}$



Calculate $\iint_S x-2y \, dS$.

Solution : Parametrize the surface S using spherical coordinates :

$$\begin{cases} \vec{r}(\theta,\phi) = \langle 2\sin(\phi)\cos(\theta), 2\sin(\phi)\sin(\theta), 2\cos(\phi) \rangle \\ 0 \leq \theta \leq 2\pi \\ 0 \leq \phi \leq \pi/2 \end{cases}$$

Calculate

$$\vec{r}_\theta = \langle -2s(\phi)s(\theta), 2s(\phi)c(\theta), 0 \rangle$$

$$\vec{r}_\phi = \langle 2c(\phi)c(\theta), 2c(\phi)s(\theta), -2s(\phi) \rangle$$

$$\vec{r}_\theta \times \vec{r}_\phi = \langle -4s^2(\phi)c(\theta), -(4s^2(\phi)s(\theta) - 0), \overbrace{-4s(\phi)c(\phi)s^2(\theta) - 4s(\phi)c(\phi)c^2(\theta)}^{= -4s(\phi)c(\phi)\overbrace{[s^2(\theta)+c^2(\theta)]}^{=1}} \rangle$$

Hence

$$\|\vec{r}_\theta \times \vec{r}_\phi\| = \sqrt{16s^4(\phi)c^2(\theta) + 16s^4(\phi)s^2(\theta) + 16s^2(\phi)c^2(\phi)}$$

$$= 4\sqrt{s^4(\phi)[c^2(\theta)+s^2(\theta)] + s^2(\phi)c^2(\phi)}$$

$$= 4\sqrt{s^2(\phi)[s^2(\phi)+c^2(\phi)]}$$

$$= 4\sin(\phi)$$

Thus, $$\iint_S x-2y \, dS = \int_0^{\pi/2}\int_0^{2\pi} [2\sin(\phi)\cos(\theta) - 4\sin(\phi)\sin(\theta)][4\sin(\phi)] \, d\theta \, d\phi$$

$$= 8\int_0^{\pi/2}\int_0^{2\pi} [\cos(\theta) - 2\sin(\theta)][\sin^2(\phi)] \, d\theta \, d\phi$$

$$= 8\int_0^{\pi/2} \underbrace{[\sin(\theta) + 2\cos(\theta)]}_{\to (0+2)-(0+2)} \sin^2(\phi) \Big|_{\theta=0}^{\theta=2\pi} d\phi$$

$$= 0$$