

(1)

Section 3.3 #113 Find unit tangent vector and unit normal of

$$\vec{r}(t) = a \cos(wt) \hat{i} + b \sin(wt) \hat{j}$$

$$\text{at } t=0. \quad = \langle a \cos(wt), b \sin(wt) \rangle$$

Soln: Compute

$$\vec{r}'(t) = \langle -aw \sin(wt), bw \cos(wt) \rangle$$

$$\begin{aligned} \|\vec{r}'(t)\| &= \sqrt{(-aw \sin(wt))^2 + (bw \cos(wt))^2} \\ &= |w| \sqrt{a^2 \sin^2(wt) + b^2 \cos^2(wt)} \end{aligned}$$

Therefore,

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \left\langle \frac{a \sin(wt)}{\sqrt{a^2 \sin^2(wt) + b^2 \cos^2(wt)}}, \frac{b \cos(wt)}{\sqrt{a^2 \sin^2(wt) + b^2 \cos^2(wt)}} \right\rangle$$

To find unit normal, first compute

$$\frac{d}{dt} (\sqrt{a^2 \sin^2(wt) + b^2 \cos^2(wt)})^{-1/2} = \left(-\frac{1}{2} (a^2 \sin^2(wt) + b^2 \cos^2(wt))^{-3/2} \right)$$

$$\begin{aligned} &\circ \left(2w a^2 \sin(wt) \cos(wt) + 2w b^2 \cos(wt) (-\sin(wt)) \right) \\ &= \frac{-w \sin(wt) \cos(wt) [a^2 + b^2]}{\sqrt{(a^2 \sin^2(wt) + b^2 \cos^2(wt))^3}} \end{aligned}$$

So, using the quotient rule,

$$\vec{T}'(t) = \frac{(\sqrt{ }) aw \cos(wt) - a \sin(wt) \left(\frac{-w \sin(wt) \cos(wt) [a^2 + b^2]}{\sqrt{ })^3} \right)}{a^2 \sin^2(wt) + b^2 \cos^2(wt)}$$

$$\frac{\sqrt{ }) (-bw \sin(wt)) - b \cos(wt) \left(\frac{-w \sin(wt) \cos(wt) [a^2 + b^2]}{\sqrt{ })^3} \right)}{a^2 \sin^2(wt) + b^2 \cos^2(wt)}$$

$$= \langle \alpha, \beta \rangle$$

Thus

$$\vec{N}(t) = \frac{\langle \alpha, \beta \rangle}{\|\langle \alpha, \beta \rangle\|} = \left\langle \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}}, \frac{\beta}{\sqrt{\alpha^2 + \beta^2}} \right\rangle$$

(2)

Thus,

$$\vec{T}(0) = \left\langle \frac{a \sin(0)}{\sqrt{a^2 \sin^2(0) + b^2 \cos^2(0)}}, \frac{b \cos(0)}{\sqrt{a^2 \sin^2(0) + b^2 \cos^2(0)}} \right\rangle \\ = \langle 0, 1 \rangle$$

and

$$\alpha|_{t=0} = \frac{\sqrt{a} w(1) - 0}{\sqrt{(0+b^2)^3}} = \frac{aw}{b^3}$$

$$\beta|_{t=0} = \frac{0 - b(1)(0)}{0+b^2} = 0$$

Thus,

$$\vec{N}(0) = \left\langle \frac{\frac{aw}{b^3}}{\sqrt{\frac{a^2 w^2}{b^6} + 0}}, 0 \right\rangle = \langle 1, 0 \rangle$$