HW9 MATH 3503 Fall 2018

Section 4.6 #264 $h(x,y) = e^x \sin(y)$

$P = (1, \frac{\pi}{2})$, $\vec{v} = -\vec{i} = \langle -1, 0 \rangle$

Soln: $\nabla h = \langle e^x \sin(y), e^x \cos(y) \rangle$

$\vec{u} = \vec{v} = \langle -1, 0 \rangle$

$D_{\vec{u}} h = \nabla h(-1,0) \cdot \vec{u}$

$= \langle e^1 \sin(\pi/2), e^1 \cos(\pi/2) \rangle \cdot \langle -1, 0 \rangle$

$= \langle e, 0 \rangle \cdot \langle -1, 0 \rangle$

$= -e$

#265 $h(x,y,z) = xyz$

$P = (2,1,1)$, $\vec{v} = 2\vec{i} + \vec{j} - \vec{k}$

Soln: $\nabla h = \langle yz, xz, xy \rangle$

$\|\vec{v}\| = \sqrt{2^2 + 1^2 + (-1)^2} = \sqrt{6}$

$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \langle \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}} \rangle$

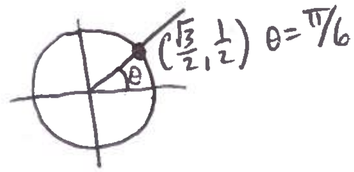
$D_{\vec{u}} h(2,1,1) = \nabla h(2,1,1) \cdot \vec{u}$

$= \langle 1, 2, 2 \rangle \cdot \langle \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}} \rangle$

$= \frac{2}{\sqrt{6}} + \frac{2}{\sqrt{6}} - \frac{2}{\sqrt{6}}$

$= \frac{2}{\sqrt{6}}$

#274 $f(x,y) = x^2 + 2y^2$, $\theta = \frac{\pi}{6}$ $\longrightarrow$



Soln:

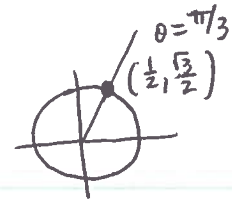
$$\nabla f = \langle 2x, 4y \rangle$$

$$\vec{u} = \langle \sqrt{3}/2, 1/2 \rangle$$

$$D_{\vec{u}} f(x,y) = \nabla f \cdot \vec{u} = \langle 2x, 4y \rangle \cdot \langle \sqrt{3}/2, 1/2 \rangle$$

$$= \sqrt{3}x + 2y$$

#279 $f(x,y) = \ln(x+2y)$, $\theta = \frac{\pi}{3}$ $\longrightarrow$

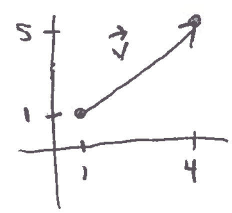


Soln: $\nabla f = \langle \frac{1}{x+2y}, \frac{2}{x+2y} \rangle$

$$\vec{u} = \langle \frac{1}{2}, \sqrt{3}/2 \rangle$$

$$D_{\vec{u}} f(x,y) = \nabla f \cdot \vec{u} = \langle \frac{1}{x+2y}, \frac{2}{x+2y} \rangle \cdot \langle \frac{1}{2}, \sqrt{3}/2 \rangle$$

$$= \frac{1}{2x+4y} + \frac{\sqrt{3}}{x+2y}$$



#284 $f(x,y) = x^2 + 3y^2$ $\quad P = (1,1)$, $Q = (4,5)$

Soln: $\nabla f = \langle 2x, 6y \rangle$

$$\vec{v} = \vec{PQ} = \langle 4,5 \rangle - \langle 1,1 \rangle = \langle 3,4 \rangle \longrightarrow \|\vec{v}\| = \sqrt{25} = 5$$

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \langle \frac{3}{5}, \frac{4}{5} \rangle$$

$$D_{\vec{u}} f(1,1) = \nabla f(1,1) \cdot \vec{u} = \langle 2,6 \rangle \cdot \langle \frac{3}{5}, \frac{4}{5} \rangle = \frac{6}{5} + \frac{24}{5} = \frac{30}{5} = 6$$

#290 $f(x,y) = xy^2 - yx^2$, $P = (-1,1)$

Soln: $\nabla f = \langle y^2 - 2yx, 2xy - x^2 \rangle$

$$\nabla f(-1,1) = \langle 1, -3 \rangle$$

#292 $f(x,y,z) = xy - \ln(z)$, $P = (2,-2,2)$

Soln: $\nabla f = \langle y, x, -\frac{1}{z} \rangle$

$$\nabla f(2,-2,2) = \langle -2, 2, -\frac{1}{2} \rangle$$

#299 maximum rate of change occurs in direction of ∇f:

$$\nabla f = \langle e^{-y}, -xe^{-y} \rangle$$

at $(1,0)$,

$$\nabla f(1,0) = \langle 1, -1 \rangle$$

The max rate of change at $(1,0)$ is

$$\|\nabla f(1,0)\| = \sqrt{2}$$

#301 direction of max rate of change:

$$\nabla f = \langle -3\sin(3x+2y), -2\sin(3x+2y) \rangle$$

at $(\frac{\pi}{6}, -\frac{\pi}{8})$ $\rightarrow$ $\nabla f(\frac{\pi}{6}, -\frac{\pi}{8}) = \langle -3\sin(3(\frac{\pi}{6})+2(-\frac{\pi}{8})), -2\sin(3(\frac{\pi}{6})+2(-\frac{\pi}{8})) \rangle$

$\frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

$$= \langle -3\sin(\tfrac{\pi}{4}), -2\sin(\pi/4) \rangle$$

$$= \langle -\frac{3\sqrt{2}}{2}, -\sqrt{2} \rangle$$

max rate of change is

$$\|\nabla f(\tfrac{\pi}{6}, -\tfrac{\pi}{8})\| = \sqrt{\frac{9(2)}{4} + 2} = \sqrt{\frac{18}{4} + \frac{8}{4}} = \sqrt{\frac{26}{4}} = \frac{\sqrt{26}}{2}$$

#306 $T(x,y,z) = \dfrac{k}{\sqrt{x^2+y^2+z^2}}$

Given: $\underbrace{120}_{\text{given}} = T(1,2,2) = \underbrace{\dfrac{k}{\sqrt{1^2+2^2+2^2}}}_{\text{calculated}} = \dfrac{k}{\sqrt{9}}$

Thus, $120 = \dfrac{k}{3} \rightarrow \boxed{k = 360}$

Therefore we have

$$T(x,y,z) = \frac{360}{\sqrt{x^2+y^2+z^2}}$$

a) vector from $(1,2,2)$ to $(2,1,3)$:

$\vec{v}$ = tip − tail $= \langle 2,1,3\rangle - \langle 1,2,2\rangle = \langle 1,-1,1\rangle$

Since $\|\vec{v}\| = \sqrt{3} \neq 1$, normalize to get

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \left\langle \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$$

Calculate

$$\nabla T = 360 \left\langle -\tfrac{1}{2}(x^2+y^2+z^2)^{-3/2}(2x), -\tfrac{1}{2}(x^2+y^2+z^2)^{-3/2}(2y), -\tfrac{1}{2}(x^2+y^2+z^2)^{-3/2}(2z) \right\rangle$$

$$= \frac{-360}{(x^2+y^2+z^2)^{3/2}} \langle x, y, z\rangle \rightarrow \nabla T(1,2,2) = \frac{-360}{(\sqrt{9})^3}\langle 1,2,2\rangle$$

$$= \frac{-360}{27}\langle 1,2,2\rangle$$

$$= -\frac{40}{3}\langle 1,2,2\rangle$$

Therefore, directional derivative is

$$D_{\vec{u}}T(1,2,2) = \vec{u} \cdot \nabla T(1,2,2)$$

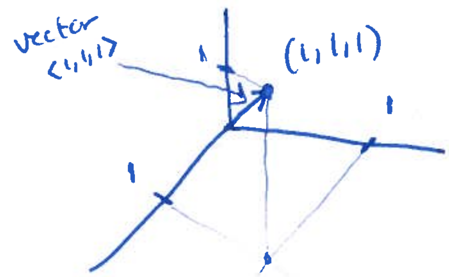
$$= \left\langle \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle \cdot \left(-\frac{40}{3}\langle 1,2,2\rangle\right)$$

$$= -\frac{40}{3}\left(\frac{1}{\sqrt{3}} \cancel{-\frac{2}{\sqrt{3}}} \cancel{+\frac{2}{\sqrt{3}}}\right)$$

$$= -\frac{40}{3\sqrt{3}}$$

b) Direction of greatest increase is

$$\nabla T = \underbrace{\frac{-360}{(x^2+y^2+z^2)^{3/2}}}_{\text{always negative}} \langle x, y, z \rangle$$



This means the direction ∇T is pointing from the point (x,y,z) towards the origin, as desired.

#308) $V(x,y) = e^{-2x}\cos(2y)$

electric intensity: $\vec{E} = -\nabla V$

a) Find $\vec{E}(\frac{\pi}{4}, 0)$:

Calculate

$$\vec{E} = -\nabla V = -\left[\langle -2e^{-2x}\cos(2y), -2e^{-2x}\sin(2y)\rangle\right]$$

$$\cancel{= \langle 2e}$$

$$= 2e^{-2x}\langle \cos(2y), \sin(2y)\rangle$$

$$\vec{E}\left(\frac{\pi}{4}, 0\right) = 2e^{-2(\pi/4)}\langle \cos(2(0)), \sin(2(0))\rangle$$

$$= 2e^{-\pi/2}\langle \cos(0), \sin(0)\rangle$$

$$= 2e^{-\pi/2}\langle 1, 0\rangle$$

b) By Theorem 4.13 (iii), electric potential V decreases fastest in direction $-\nabla V$, but this is precisely the definition of $\vec{E}$!

Section 4.7

#318 $f(x,y) = -x^3 + 4xy - 2y^2 + 1$

$$\begin{cases} f_x = -3x^2 + 4y \overset{\text{set}}{=} 0 & (i) \\ f_y = 4x - 4y \overset{\text{set}}{=} 0 & (ii) \end{cases}$$

Solve (ii) for x to get

(*) $x = y$

Plug into (i) to get

$$-3y^2 + 4y = 0$$

$$y(-3y+4) = 0$$

$$y = 0, \quad y = \frac{4}{3}$$

$$\downarrow (*) \qquad \downarrow (*)$$

$$x = 0 \qquad x = 4/3$$

So our critical pts are

$(0,0)$ and $(4/3, 4/3)$.

Calculate $f_{xx} = 6x$, $f_{yy} = -4$, $f_{xy} = 4$

and $D = f_{xx} f_{yy} - (f_{xy})^2 = -24x - 4^2 = -24x - 16$

Analyze c.p. $(0,0)$

$D(0,0) = -16 < 0$

$\downarrow$

saddle pt at $(0,0)$

Analyze c.p. $(4/3, 4/3)$

$D(4/3, 4/3) = -24(4/3) - 16$

$= -96/3 - 16$

$= -32 - 16 < 0$

$f_{xx}(4/3, 4/3) = 6(4/3) = 8 > 0$

$\downarrow$

local min at $(4/3, 4/3)$

#320 $f(x,y) = x^2 - 6x + y^2 + 4y - 8$

$$\begin{cases} f_x = 2x - 6 \overset{\text{set}}{=} 0 & (i) \\ f_y = 2y + 4 \overset{\text{set}}{=} 0 & (ii) \end{cases}$$

$$\boxed{f_{xx} = 2 \quad f_{yy} = 2 \quad f_{xy} = 0 \longrightarrow D = f_{xx}f_{yy} - (f_{xy})^2 = 4}$$

From (i), $x = 3$. From (ii), $y = -2$.

Therefore only c.p. is at $(3, -2)$.

Calculate $D(3,-2) = 4 > 0$ and $f_{xx}(3,-2) = 2 > 0$. Thus local min at $(3,-2)$.

#322 $f(x,y) = 8xy(x+y) + 7 \overset{\text{expand}}{=} 8x^2y + 8xy^2 + 7$

$$\begin{cases} f_x = 16xy + 8y^2 \overset{\text{set}}{=} 0 & (i) \\ f_y = 8x^2 + 16xy \overset{\text{set}}{=} 0 & (ii) \end{cases}$$

$$\boxed{f_{xx} = 16y \quad f_{yy} = 16x \quad f_{xy} = 16x + 16y \rightarrow D = (16x)(16y) - (16x + 16y)^2}$$

[illegible]

From (i),

$$y(16x + 8y) = 0$$

$y = 0$

$\downarrow$ plug into (ii)

$8x^2 = 0$

$\downarrow$

$x = 0$

$\downarrow$

$(0,0)$ a c.p.

$16x + 8y = 0$.

$\downarrow$

$y = \frac{-16x}{8} = -2x$

$\downarrow$ plug into (ii)

$8x^2 + 16x(-2x) = 0 \rightarrow 8x^2 - 32x^2 = 0$

$\rightarrow -26x^2 = 0$

$\rightarrow x^2 = 0$

$\rightarrow x = 0$

$\Rightarrow y = -2(0) = 0$

$\Rightarrow (0,0)$ is a c.p.

Analyze only c.p. $(0,0)$:

$$D(0,0) = 0 - 0 = 0 \quad \text{inconclusive!}$$

#324

$$f(x,y) = x^3 + y^3 - 300x - 75y - 3$$

$$\begin{cases} f_x = 3x^2 - 300 \overset{set}{=} 0 & (i) \\ f_y = 3y^2 - 75 \overset{set}{=} 0 & (ii) \end{cases}$$

$$\boxed{f_{xx} = 6x, \quad f_{yy} = 6y, \quad f_{xy} = 0 \longrightarrow D = 36xy}$$

From (i),

$$x^2 = 100 \rightarrow x = \pm 10$$

From (ii)

$$y^2 = 25 \rightarrow y = \pm 5$$

Critical pts are $(10,5)$; $(10,-5)$, $(-10,5)$, $(-10,-5)$

Analyze c.p.'s

$\left[\begin{array}{l} D(10,5) = 36(10)(5) > 0 \\ f_{xx}(10,5) = 6(10) > 0 \end{array}\right] \longrightarrow$ local min at $(10,5)$

$\left[D(10,-5) = 36(10)(-5) < 0 \right] \longrightarrow$ saddle pt at $(10,-5)$

$\left[D(-10,5) = 36(-10)(5) < 0 \right] \longrightarrow$ saddle pt at $(-10,5)$

$\left[\begin{array}{l} D(-10,-5) = 36(-10)(-5) > 0 \\ f_{xx}(-10,-5) = 6(-10) < 0 \end{array}\right] \longrightarrow$ local max at $(-10,-5)$