HW 13 MATH 3503 FALL 2018

§6.2 | #44 | C given by $\begin{cases} \vec{r}(t) = \langle t, (1-t), 0 \rangle \\ 0 \le t \le 1 \end{cases}$ $\longrightarrow$ $\vec{r}\,' = \langle 1, -1, 0 \rangle$

$\|\vec{r}\,'\| = \sqrt{2}$

Calculate

$$\int_C x+y \, ds = \int_0^1 (t + (1-t)) \sqrt{2} \, dt$$

$$= \sqrt{2} \int_0^1 1 \, dt$$

$$= \sqrt{2}$$

#46 | C given by $\begin{cases} \vec{r}(t) = \langle \sin(t), \cos(t), 8t \rangle \\ 0 \le t \le \frac{\pi}{2} \end{cases}$ $\longrightarrow$ $\vec{r}\,' = \langle \cos(t), -\sin(t), 8 \rangle$

$\|\vec{r}\,'\| = \sqrt{65}$

Calculate

$$\int_C x^2 + y^2 + z^2 \, ds = \int_0^{\pi/2} (\sin^2(t) + \cos^2(t) + 64t^2) \sqrt{65} \, dt$$

$$= \sqrt{65} \int_0^{\pi/2} 1 + 64t^2 \, dt$$

$$= \sqrt{65} \left[t + \frac{64}{3} t^3 \right|_0^{\pi/2}$$

$$= \sqrt{65} \left[\frac{\pi}{2} + \underbrace{\frac{64\pi^3}{24}} \right]$$

$$= \frac{32\pi^3}{12}$$

$$= \frac{16\pi^3}{6}$$

$$= \frac{8\pi^3}{3}$$

#53) $\vec{F} = \langle 2y, 3x, x+y \rangle$

$$\begin{cases} \vec{r}(t) = \langle \cos(t), \sin(t), \frac{1}{6} \rangle \longrightarrow \vec{r}\,' = \langle -\sin(t), \cos(t), 0 \rangle \\ 0 \le t \le 2\pi \end{cases}$$

$$\text{Work} = \int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \langle 2\sin(t), 3\cos(t), \cos(t)+\sin(t) \rangle \cdot \langle -\sin(t), \cos(t), 0 \rangle \, dt$$

$$= \int_0^{2\pi} -2\underbrace{\sin^2(t)}_{=\frac{1-\cos(2t)}{2}} + 3\underbrace{\cos^2(t)}_{=\frac{1+\cos(2t)}{2}} \, dt$$

$$= \int_0^{2\pi} (-1+\cos(2t)) + \frac{3}{2}(1+\cos(2t)) \, dt$$

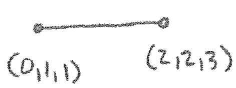
$$= \int_0^{2\pi} \frac{1}{2} + \frac{5}{2}\cos(2t) \, dt$$

$$= \left[\frac{1}{2}t + \frac{5}{4}\sin(2t)\right|_0^{2\pi}$$

$$= \left[\frac{1}{2}(2\pi) + \frac{5}{4}\underbrace{\cancel{\sin(4\pi)}}_{=0}\right] - [0+0]$$

$$= \pi$$

#58) C:



$$\Rightarrow \begin{cases} \vec{r}(t) = t\langle 2,2,3 \rangle + (1-t)\langle 0,1,1 \rangle \\ \quad = \langle 2t, 2t, 3t \rangle + \langle 0, 1-t, 1-t \rangle \\ \quad = \langle 2t, 1+t, 1+2t \rangle \\ 0 \le t \le 1 \end{cases} \rightarrow \vec{r}\,' = \langle 2,1,2 \rangle$$

$$\|\vec{r}\,'\| = \sqrt{9} = 3$$

Calculate

$$\int_C y\,ds = \int_0^1 (1+t)(3)\,dt = 3\int_0^1 1+t\,dt$$

$$= 3\left[t + \frac{t^2}{2}\right|_0^1$$

$$= 3\left[(1+\tfrac{1}{2}) - 0\right]$$

$$= 3\left(\tfrac{3}{2}\right) = \frac{9}{2}$$

#67) $\vec{F} = \langle x, y, -5z \rangle$

$\begin{cases} \vec{r}(t) = \langle 2\cos(t), 2\sin(t), t \rangle \rightarrow \vec{r}\,' = \langle -2\sin(t), 2\cos(t), 1 \rangle \\ 0 \le t \le 2\pi \end{cases}$

$$\text{Work} = \int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \langle 2\cos(t), 2\sin(t), -5t \rangle \cdot \langle -2\sin(t), 2\cos(t), 1 \rangle \, dt$$

$$= \int_0^{2\pi} \underbrace{-4\cos(t)\sin(t) + 4\sin(t)\cos(t)}_{0} - 5t \, dt$$

$$= \int_0^{2\pi} -5t \, dt$$

$$= -\frac{5}{2}t^2 \Big|_0^{2\pi} = -\frac{5}{2}(4\pi) = -10\pi$$

§6.3) #106) $\vec{F} = \langle P, Q \rangle = \langle 2xy^3, 3y^2x^2 \rangle$. By Thm 6.10, compute

$$\left.\begin{array}{l} 6xy^2 = P_y \overset{?\checkmark}{=} Q_x = 6y^2x \\ \cancel{= P_z \overset{?}{=} R_x =} \leftarrow \text{no } z\text{'s} + R=0\text{, ignore} \\ \cancel{Q_z \overset{?}{=} R_y =} \leftarrow \text{no } z\text{'s} + R=0\text{, ignore} \end{array}\right\} \longrightarrow \vec{F} \text{ is conservative}$$

Find potential function

$$\underbrace{\langle 2xy^3, 3y^2x^2 \rangle}_{\text{given}} = \vec{F} \underset{\substack{\uparrow \\ \text{def of} \\ \text{conservative}}}{=} \nabla f \underset{\substack{\uparrow \\ \text{def of } \nabla f}}{=} \langle f_x, f_y \rangle \Rightarrow \begin{cases} f_x = 2xy^3 \xrightarrow{\int \cdots dx} f = x^2y^3 + g(y) \\ f_y = 3y^2x^2 \end{cases}$$

arbitrary function depends only on y

$$\underbrace{3y^2x^2}_{\text{given}} = f_y = \underbrace{3x^2y^2 + g'(y)}_{\text{computed}}$$

$$\Downarrow$$

$$0 = g'(y)$$

$$\Downarrow$$

$g(y) = C$ for some constant

Therefore, any function of the form

$$f(x,y) = x^2y^3 + C$$

is a potential function for $\vec{F}$.

(note: check by computing ∇f!)

#107) $\vec{F} = \langle \overbrace{-y+e^x \sin y}^{P}, \overbrace{(x+2)e^x \cos(y)}^{Q} \rangle$

Thm 6.11 : $-1+e^x\cos(y) = P_y \overset{?\times}{=} Q_x = e^x\cos(y) + (x+2)e^x\cos(y)$

Thus $\vec{F}$ is not conservative.

#112) Evaluate $\int_C \overbrace{\langle y, x\rangle}^{\vec{F}} \cdot d\vec{r}$ where C is any path from $(0,0)$ to $(2,4)$.

Soln : Find the potential for $\vec{F}$:

$$\langle y, x\rangle = \vec{F} = \nabla f = \langle f_x, f_y\rangle \Rightarrow \begin{cases} f_x = y \xrightarrow{\int \cdots dx} f = xy + g(y) \\ f_y = x \end{cases}$$

$$\underbrace{x}_{\text{given}} = f_y = x + g'(y)$$

$$\Downarrow$$

$$0 = g'(y)$$

$$\Downarrow \int \cdots dy$$

$$g(y) = C$$

(don't need the +C)

Therefore, $f = xy$ is a potential of $\vec{F}$.

By the Fundamental thm of line integrals (Thm 6.7),

$$\int_C \langle y, x\rangle \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(2,4) - f(0,0)$$
$$= 2(4) - 0(0)$$
$$= 8$$

#116) $\vec{F} = \langle \overbrace{12xy}^{P}, \overbrace{6(x^2+y^2)}^{Q} \rangle$

$12x = P_y \overset{?\checkmark}{=} Q_x = 12x \Rightarrow \vec{F}$ conservative

Find potential

$$\langle 12xy, 6(x^2+y^2)\rangle = \vec{F} = \nabla f = \langle f_x, f_y\rangle \Rightarrow \begin{cases} f_x = 12xy \xrightarrow{\int \cdots dx} \boxed{f = 6x^2y + g(y)} \ (*) \\ f_y = 6(x^2+y^2) \end{cases}$$

$$\underbrace{6x^2+6y^2}_{\text{given}} = f_y = 6x^2 + g'(y)$$

$$\Downarrow$$

$$6y^2 = g'(y) \overset{\int \cdots dy}{\Longrightarrow} \boxed{g(y) = 2y^3 + C} \ (**)$$

Therefore, using (*) and (**) the potential is

$$f = 6x^2y + 2y^3 + C \quad \text{for any } C \in \mathbb{R}.$$

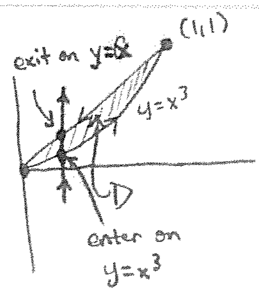
#126 | Evaluate $\int_C \nabla f \cdot d\vec{r}$ where $f = \cos(\pi x) + \sin(\pi y) - xyz$ and C is any path from $(1, \frac{1}{2}, 2)$ to $(2, 1, -1)$.

Soln: By FTLI,

$$\int_C \nabla f \cdot d\vec{r} = f(2,1,-1) - f(1,\tfrac{1}{2},2)$$
$$= \left(\cos(2\pi) + \sin(\pi) - 2(1)(-1)\right) - \left(\cos(\pi) + \sin(\pi/2) - (1)(\tfrac{1}{2})(2)\right)$$
$$= (1 + 0 + 2) - (-1 + 1 - 1)$$
$$= 3 + 1 = 4$$

§6.4 | #146 | Evaluate $\int_C 2xy\,dx + (x+y)\,dy$ where C is path from $(0,0)$ to $(1,1)$ along $y = x^3$ and from $(1,1)$ to $(0,0)$ along $y = x$ oriented counterclockwise

Soln:



Calculate

$$\int_C 2xy\,dx + (x+y)\,dy = \int_C \langle \overset{P}{2xy}, \overset{Q}{x+y} \rangle \cdot d\vec{r} \longrightarrow Q_x - P_y = (1+0) - 2x = 1 - 2x$$

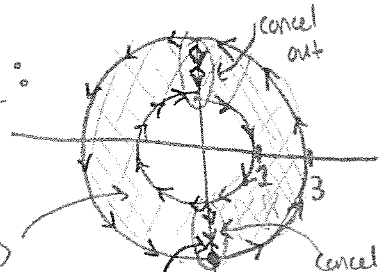
Green's theorem Thm 6.12

$$= \iint_D \overbrace{1 - 2x}^{Q_x - P_y}\, dA$$
$$= \int_0^1 \int_{x^3}^{x} 1 - 2x \; dy\,dx$$
$$= \int_0^1 y - 2xy \Big|_{y=x^3}^{y=x} dx$$
$$= \int_0^1 (x - 2x^2) - (x^3 - 2x^4)\,dx$$
$$= \int_0^1 2x^4 - x^3 - 2x^2 + x\,dx = \frac{2}{5} - \frac{1}{4} - \frac{2}{3} + \frac{1}{2} = -\frac{1}{60}$$

#150 $\int_C xy\,dx + (x+y)dy$ where C is boundary of region between $x^2+y^2=1$ and $x^2+y^2=9$ oriented counterclockwise

6

Soln:



$D = \{(r,\theta): 1 \le r \le 3,\ 0 \le \theta \le 2\pi\}$

draw C like in Figure 6.45

↑ union of green and blue parts

Calculate

$$\int_C xy\,dx + (x+y)dy = \int_C \langle \overset{P}{xy}, \overset{Q}{x+y} \rangle \cdot d\vec{r} \qquad Q_x - P_y = 1-x$$

Green's thm

$$= \iint_D 1-x\,dA$$

extra

$$= \int_0^{2\pi}\int_1^3 (1-r\cos(\theta))\,r\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{r^2}{2} - \frac{r^3}{3}\cos(\theta)\Big|_{r=1}^{r=3}\,d\theta$$

$$= \int_0^{2\pi} \left(\frac{9}{2} - \frac{27}{3}\cos(\theta)\right) - \left(\frac{1}{2} - \frac{1}{3}\cos(\theta)\right) d\theta$$

$$= \int_0^{2\pi} 4 - \frac{26}{3}\cos(\theta)\,d\theta$$

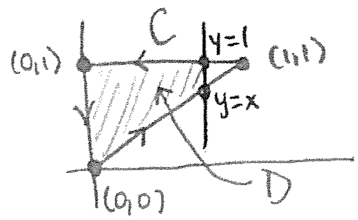
$$= 4\theta - \frac{26}{3}\sin(\theta)\Big|_0^{2\pi}$$

$$= (8\pi - 0) - (0 - 0)$$

$$= 8\pi$$

#152 Calculate $\int_C xy\,dx + \sqrt{y^2+1}\,dy$ where C is curve of line segments from $(0,0)$ to $(1,1)$ to $(0,1)$ and back to $(0,0)$. (7)

Soln:



Calculate

$$Q_x - P_y = 0 - x = -x$$

$$\int_C xy\,dx + \sqrt{y^2+1}\,dy = \int_C \langle \underset{P}{xy}, \underset{Q}{\sqrt{y^2+1}} \rangle \cdot d\vec{r}$$

Green's thm

$$= \iint_D -x\,dA$$

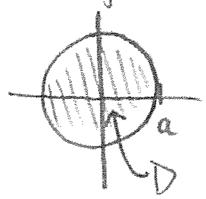
$$= \int_0^1 \int_x^1 -x\,dy\,dx$$

$$= \int_0^1 -xy \Big|_{y=x}^{y=1} dx$$

$$= \int_0^1 -x + x^2\,dx$$

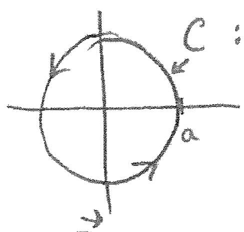
$$= -\frac{1}{2} + \frac{1}{3} = -\frac{1}{6}$$

#166 Area of disk of radius a is $\iint_D 1\, dA$ where D is given by



To use Green's thm, we must parametrize the boundary circle of D and find a vector field $\vec{F} = \langle P, Q \rangle$ so that $Q_x - P_y = 1$.

Parametrize bdy(D)



$C: \vec{r}(t) = \langle a\cos t, a\sin t \rangle \rightsquigarrow \vec{r}\,'(t) = \langle -a\sin(t), a\cos(t) \rangle$

$0 \le t \le 2\pi$

Find an $\vec{F}$

We want $Q_x - P_y = 1$

Can accomplish this if $\left.\begin{array}{l} Q_x = \frac{1}{2} \xrightarrow{\int \cdots dy} Q = \frac{x}{2} \\ P_y = -\frac{1}{2} \xrightarrow[\int \cdots d]{} P = -\frac{y}{2} \end{array}\right\} \to \vec{F} = \left\langle -\frac{y}{2}, \frac{x}{2} \right\rangle$

Therefore,

$$\text{Area}_{\text{Disk}} = \iint_D 1\, dA$$

$$\xrightarrow{\text{Green's}} = \oint_C \left\langle -\frac{y}{2}, \frac{x}{2} \right\rangle \cdot d\vec{r}$$

$$\xrightarrow{(6.9),\ \text{pg. } 678} = \int_0^{2\pi} \left\langle \frac{-a\sin t}{2}, \frac{a\cos t}{2} \right\rangle \cdot \langle -a\sin t, a\cos t \rangle\, dt$$

$$= \int_0^{2\pi} \frac{a^2\sin^2(t)}{2} + \frac{a^2\cos^2(t)}{2}\, dt$$

$$= \frac{a^2}{2}\int_0^{2\pi} 1\, dt = \left(\frac{a^2}{2}\right)(2\pi) = \pi a^2,$$

as was to be shown.

§6.5

#212) $\vec{F} = \langle xy^2z^4,\ 2x^2y+z,\ y^3z^2 \rangle$

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2z^4 & 2x^2y+z & y^3z^2 \end{bmatrix}$$

$$= \left\langle 3y^2z^2-1,\ -\left(0-4xy^2z^3\right),\ 4xy-2xyz^4 \right\rangle$$

$$= \langle 3y^2z^2-1,\ 4xy^2z^3,\ 4xy-2xyz^4 \rangle$$

#222) $\vec{F} = \langle x^2z,\ y^2x,\ y+2z \rangle$

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle x^2z,\ y^2x,\ y+2z \rangle$$

$$= \frac{\partial}{\partial x}(x^2z) + \frac{\partial}{\partial y}(y^2x) + \frac{\partial}{\partial z}(y+2z)$$

$$= 2xz + 2yx + 2$$