HW12 MATH 3503 FALL 2018

#181 $$\iiint_B 2x+3y^2+4z^3\,dV=\int_0^1\int_0^2\int_0^3 2x+3y^2+4z^3\,dz\,dy\,dx$$

$$=\int_0^1\int_0^2 2xz+3y^2z+z^4\Big|_{z=0}^{z=3}\,dy\,dx$$

$$=\int_0^1\int_0^2 6x+9y^2+81\,dy\,dx$$

$$=\int_0^1 6xy+3y^3+81y\Big|_{y=0}^{y=2}\,dx$$

$$=\int_0^1 12x+24+162\,dx$$

$$=6x^2+186x\Big|_0^1$$

$$=6+186=192$$

#184 $$\iiint_B z\sin(x)+y^2\,dV=\int_0^{\pi}\int_0^1\int_{-1}^2 z\sin(x)+y^2\,dz\,dy\,dx$$

$$=\int_0^{\pi}\int_0^1 \frac{z^2}{2}\sin(x)+y^2z\Big|_{z=-1}^{z=2}\,dy\,dx$$

$$=\int_0^{\pi}\int_0^1 \frac{3}{2}\sin(x)+3y^2\,dy\,dx$$

$$=\int_0^{\pi}\frac{3}{2}\sin(x)y+y^3\Big|_{y=0}^{y=1}\,dx$$

$$=\int_0^{\pi}\frac{3}{2}\sin(x)+1\,dx=\frac{3}{2}(-\cos(x))+x\Big|_{x=0}^{x=\pi}$$

$$=\left[-\frac{3}{2}(\cos(\pi))+\pi\right]-\left[-\frac{3}{2}\cos(0)+0\right]$$

$$=\frac{3}{2}+\pi+\frac{3}{2}=3+\pi$$

#196 $\iiint_E x^3+y^3+z^3\,dV = \int_0^2 \int_0^{2x} \int_0^{4-x-y} x^3+y^3+z^3\,dz\,dy\,dx$

$= \int_0^2 \int_0^{2x} zx^3+zy^3+\frac{z^4}{4}\Big|_{z=0}^{z=4-x-y} dy\,dx$

$= \int_0^2 \int_0^{2x} \underbrace{(4-x-y)(x^3+y^3)+\frac{1}{4}(4-x-y)^4}_{\text{expand}}\,dy\,dx$

$= \int_0^2 \int_0^{2x} -\frac{3}{4}x^4+\frac{3}{2}x^2y^2-12x^2y+24x^2-12xy^2+48xy-64x-\frac{3}{4}y^4+24y^2-64y+64\,dy\,dx$

24
4
96

$= \int_0^2 -\frac{3}{4}yx^4+\frac{1}{2}x^2y^3-6x^2y^2+24x^2y-4xy^3+24xy^2-64xy-\frac{3}{20}y^5+8y^3-32y^2+64y\Big|_{y=0}^{y=2x} dx$

64
2
128

$= \int_0^2 -\frac{3}{2}x^5+4x^5-24x^4+48x^3-32x^4+96x^3-128x^2-\frac{96}{20}x^5+64x^3-128x^2+128x\,dx$

32
3
96
32
4
128

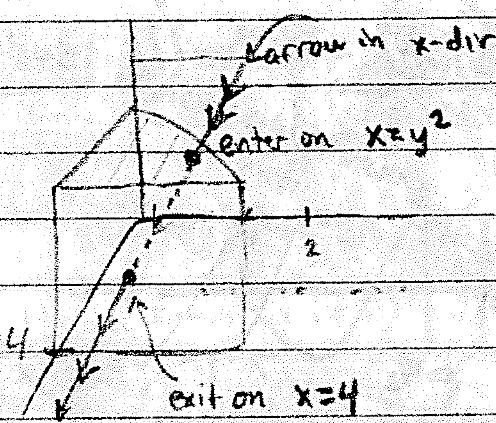
64
2
128

simplify $= \int_0^2 -\frac{23}{10}x^5-56x^4+208x^3-256x^2+128x\,dx$

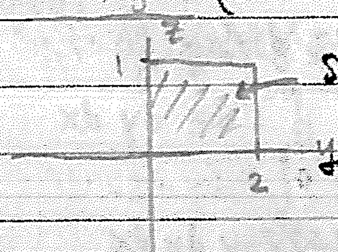
$= -\frac{23}{60}x^6-\frac{56}{5}x^5+52x^4-\frac{256}{3}x^3+64x^2\Big|_0^2$

$= -\frac{23}{60}(64)-\frac{56}{5}(32)+52(16)-\frac{256}{3}(8)+64(4)$

#212



Shadow region (smash region against arrow into yz-plane)



$S=\left\{(y,z): \begin{matrix} 0\le y\le 2 \\ 0\le z\le 1\end{matrix}\right\}$

$$\iiint_E xyz\,dV = \int_0^1\int_0^2\int_{y^2}^4 xyz\,dx\,dy\,dz$$

$$= \int_0^1\int_0^2 \frac{x^2yz}{2}\Big|_{x=y^2}^{x=4} dy\,dz$$

$$= \int_0^1\int_0^2 8yz - \frac{1}{2}y^5 z\,dy\,dz$$

$$= \int_0^1 4y^2z - \frac{1}{12}y^6 z\Big|_{y=0}^{y=2} dz$$

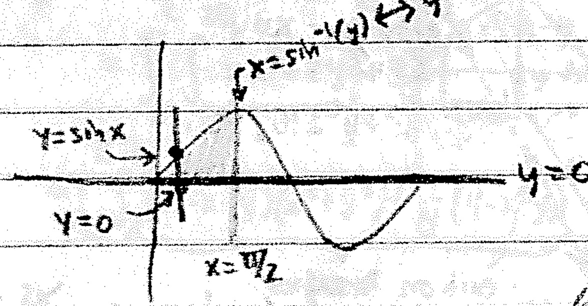
$$= \int_0^1 \underbrace{8z - \frac{16}{3}z}_{8-\frac{16}{3}=\frac{24-16}{3}=\frac{8}{3}}\,dz = \int_0^1 \frac{8}{3}z\,dz = \frac{8}{3}\cdot\frac{1}{2}z^2\Big|_0^1 = \frac{8}{6} = \frac{4}{3}$$

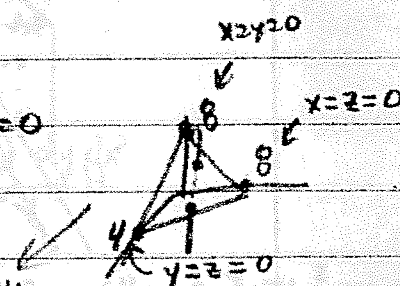
$\frac{64}{12} = \frac{32}{6} = \frac{16}{3}$

$24 - 16 = 8$

"projection"

#224 Shadow region





Plane: $2x+y+z=8 \Rightarrow z=8-2x-y$

So,

$$\text{Vol}(E)=\iiint_E 1\,dV=\int_0^{\pi/2}\int_0^{\sin x}\int_0^{8-2x-y} 1\,dz\,dy\,dx$$

$$=\int_0^{\pi/2}\int_0^{\sin x} 8-2x-y\,dy\,dx$$

$$=\int_0^{\pi/2} 8y-2xy-\frac{y^2}{2}\Big|_0^{\sin x}\,dx$$

$$=\int_0^{\pi/2} 8\sin(x)-2x\sin(x)-\frac{\sin^2 x}{2}\,dx$$

$\int$ by parts

use $\sin^2(x)=\frac{1-\cos(2x)}{2}$

$= \cdots$

$= 6-\frac{\pi}{8}$

§5.5 #242 $\iiint_E xz^2\,dV = \iint_S \int_{-1}^{1} xz^2\,dz\,dA$

given by $x^2+y^2 \le 16,\ x \ge 0,\ y \le 0$

$\Downarrow$

$S = \left\{(r,\theta) : \begin{array}{l} 0 \le r \le 4 \\ -\pi/2 \le \theta \le 0 \end{array}\right\}$

Therefore,

$$\iiint_E xz^2\,dV = \int_{-\pi/2}^{0}\int_{0}^{4}\int_{-1}^{1} (r\cos(\theta))\,z^2 \cdot \overset{\text{extra}}{r}\,dz\,dr\,d\theta$$

$$= \int_{-\pi/2}^{0}\int_{0}^{4} r^2\cos(\theta)\,\frac{z^3}{3}\Big|_{-1}^{1}\,dr\,d\theta$$

$$= \frac{2}{3}\int_{-\pi/2}^{0}\int_{0}^{4} r^2\cos(\theta)\,dr\,d\theta$$

$$= \frac{2}{3}\int_{-\pi/2}^{0} \frac{r^3}{3}\cos(\theta)\Big|_{r=0}^{r=4}\,d\theta$$

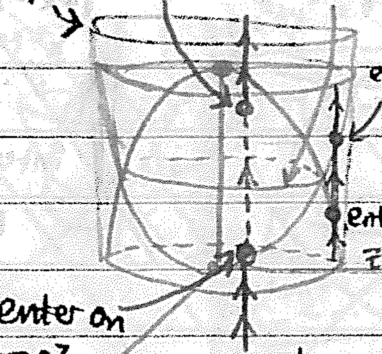
$$= \frac{2}{3}\left(\frac{64}{3}\right)\int_{-\pi/2}^{0}\cos(\theta)\,d\theta$$

$$= \frac{128}{9}\sin(\theta)\Big|_{-\pi/2}^{0} = \frac{128}{9}(0-(-1)) = \frac{128}{9}$$

cylinder exit on $z=1-r^2$

circle of intersection

#26 ||



exit on $z=r^2$

enter on $z=1-r^2$

enter on $z=r^2$

$$z = 1-x^2-y^2 = 1-r^2 \quad \leftarrow \text{polar}$$

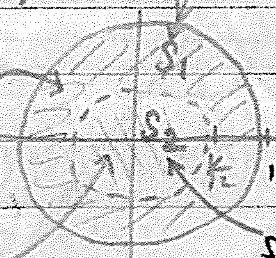
$$z = x^2+y^2 = r^2 \quad \leftarrow \text{polar}$$

$$S_1 = \left\{(r,\theta): \begin{matrix} \frac{1}{\sqrt{2}} \le r \le 1 \\ 0 \le \theta \le 2\pi \end{matrix}\right\}$$

<u>circle of $\cap$</u>

$$1-r^2 = r^2$$

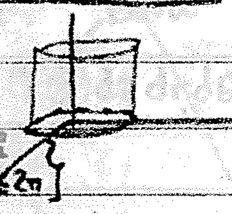
$$1 = 2r^2 \rightarrow r = \frac{1}{\sqrt{2}}$$



here, $1-r^2$ on bottom, r^2 on top

<u>here</u> r^2 on bottom, $1-r^2$ on top

<u>cylinder $x^2+y^2=1$</u>



$$S_2 = \left\{(r,\theta): \begin{matrix} 0 \le r \le \frac{1}{\sqrt{2}} \\ 0 \le \theta \le 2\pi \end{matrix}\right\}$$

$$Vol(E) = \iiint_E 1\, dV = \iint_{S_1} \int_{1-x^2-y^2}^{x^2+y^2} 1\, dz\, dA + \iint_{S_2} \int_{x^2+y^2}^{1-x^2-y^2} 1\, dz\, dA$$

$$= \int_0^{2\pi}\int_{1/\sqrt{2}}^{1}\int_{1-r^2}^{r^2} \underset{\text{extra}}{r}\, dz\, dr\, d\theta + \int_0^{2\pi}\int_0^{1/\sqrt{2}}\int_{r^2}^{1-r^2} \underset{\text{extra}}{r}\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_{1/\sqrt{2}}^{1} rz \Big|_{1-r^2}^{r^2} dr\, d\theta + \int_0^{2\pi}\int_0^{1/\sqrt{2}} rz \Big|_{r^2}^{1-r^2} dr\, d\theta$$

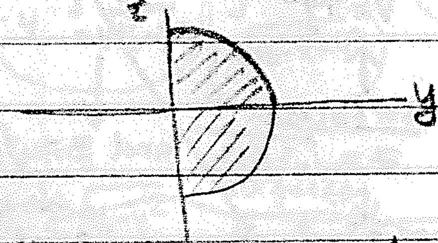
$$= \int_0^{2\pi}\int_{1/\sqrt{2}}^{1} \underbrace{r(r^2) - r(1-r^2)}_{= 2r^3 - r} dr\, d\theta + \int_0^{2\pi}\int_0^{1/\sqrt{2}} \underbrace{r(1-r^2) - r(r^2)}_{= r - 2r^3} dr\, d\theta$$

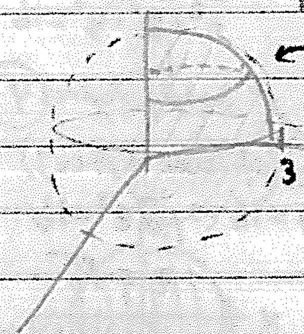
$$= \int_0^{2\pi} \frac{1}{2}r^4 - \frac{r^2}{2}\Big|_{1/\sqrt{2}}^{1} d\theta + \int_0^{2\pi} \frac{r^2}{2} - \frac{1}{2}r^4 \Big|_0^{1/\sqrt{2}} d\theta$$

$$= \int_0^{2\pi} \left(0 - \left(\frac{1}{8} - \frac{1}{4}\right)\right) d\theta + \int_0^{2\pi} \left(\frac{1}{4} - \frac{1}{8}\right) - 0\, d\theta$$

$$= \int_0^{2\pi} \frac{1}{8}\, d\theta + \int_0^{2\pi} \frac{1}{8}\, d\theta = \frac{2\pi}{8} + \frac{2\pi}{8} = \frac{\pi}{2}$$

#270 Shadow region in yz-plane (set $x=0$)





← piece of sphere $x^2+y^2+z^2=9$ where $y\geq 0,\ z\geq 0$

$$B=\left\{(x,y,z): x^2+y^2+z^2\leq 9,\ y\geq 0,\ z\geq 0\right\}$$

$$=\left\{(\rho,\theta,\phi): 0\leq\rho\leq 3,\ 0\leq\theta\leq\pi,\ 0\leq\phi\leq\pi/2\right\}$$

$$\iiint_B 1-\sqrt{x^2+y^2+z^2}\,dV=\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{3}(1-\sqrt{\rho^2})(\rho^2\sin(\phi))\,d\rho\,d\theta\,d\phi$$

↑ extra ↖ integrand $(\rho^2-\rho^3)\sin(\phi)$

$$=\int_0^{\pi/2}\int_0^{\pi}\left(\frac{\rho^3}{3}-\frac{\rho^4}{4}\right)\sin(\phi)\Big|_{\rho=0}^{\rho=3}d\theta\,d\phi$$

$$=\int_0^{\pi/2}\int_0^{\pi}\left(9-\frac{81}{4}\right)\sin(\phi)\,d\theta\,d\phi$$

$$=\left(\frac{36-81}{4}\right)\int_0^{\pi/2}\int_0^{\pi}\sin(\phi)\,d\theta\,d\phi=\pi\left(\frac{-45}{4}\right)\int_0^{\pi/2}\sin(\phi)\,d\phi$$

$$=-\frac{45\pi}{4}\left[-\cos(\phi)\right]_0^{\pi/2}$$

$$=-\frac{45\pi}{4}\left[0-\underbrace{(-\cos(0))}_{=1}\right]$$

$$=-\frac{45\pi}{4}$$

#284 $\displaystyle\int_{-4}^{4}\int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}}\int_{-\sqrt{16-x^2-y^2}}^{\sqrt{16-x^2-y^2}} x^2+y^2+z^2 \, dz\, dx\, dy = I$

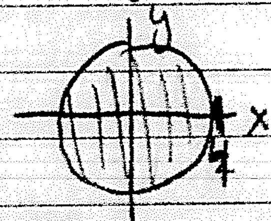
curves

$x = \pm\sqrt{16-y^2}$

$\updownarrow$

$x^2+y^2=16$

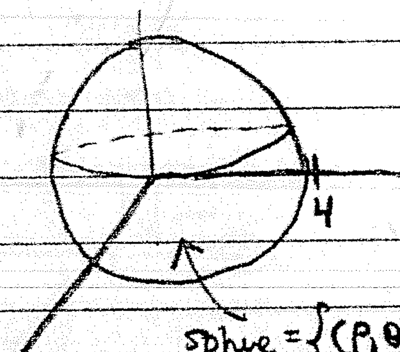
$\Downarrow$



surfaces

$z = \pm\sqrt{16-x^2-y^2} \longleftrightarrow x^2+y^2+z^2=16$

sphere radius 4



$$\text{sphere} = \left\{(\rho,\theta,\phi) : \begin{array}{l} 0 \le \rho \le 4 \\ 0 \le \theta \le 2\pi \\ 0 \le \phi \le \pi \end{array}\right\}$$

So,

$$I = \int_0^{\pi}\int_0^{2\pi}\int_0^{4} (\rho^2)\,(\rho^2 \sin(\phi))\, d\rho\, d\theta\, d\phi$$

extra

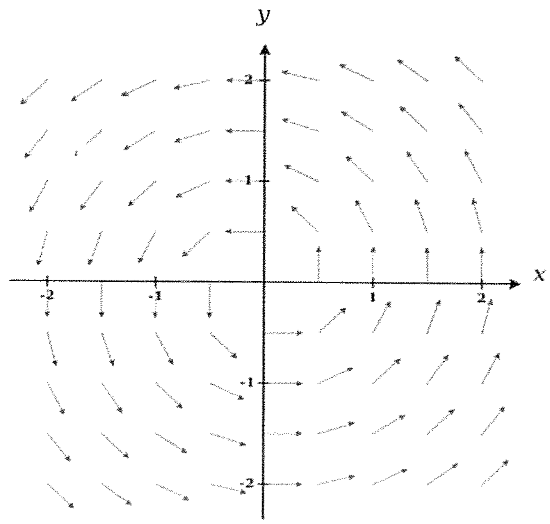
$$= \int_0^{\pi}\int_0^{2\pi} \frac{\rho^5}{5}\sin(\phi)\Big|_0^4 \, d\theta\, d\phi$$

$$= \int_0^{\pi}\int_0^{2\pi} \frac{4^5}{5}\sin(\phi)\, d\theta\, d\phi = \frac{4^5}{5}\cdot 2\pi \int_0^{\pi} \sin(\phi)\, d\phi$$

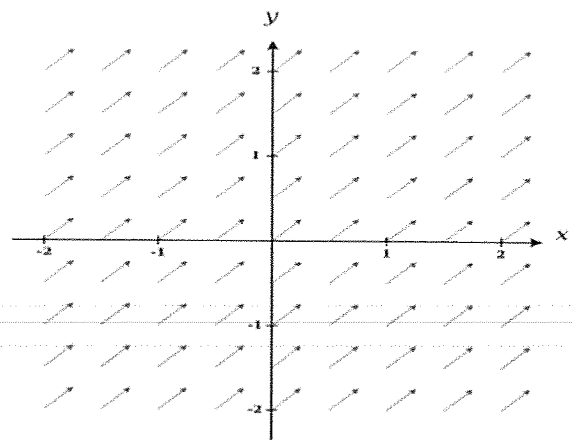
$$= \frac{4^5}{5}\cdot 2\pi \underbrace{\left[-\cos(\phi)\Big|_0^{\pi}\right]}_{=}$$

$$= \frac{4^5}{5}\cdot 2\pi \cdot 2$$

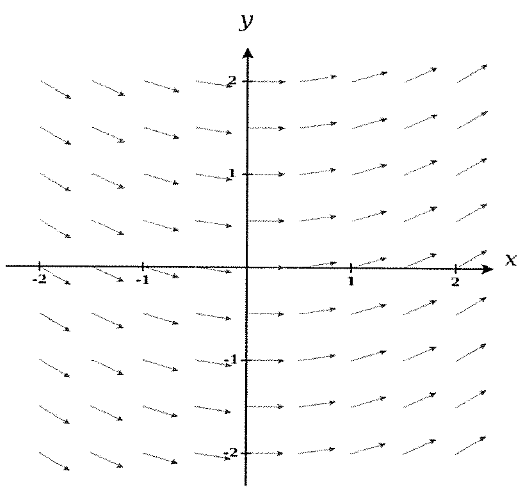
Section 6.1, Problem 6:



Section 6.1, Problem 8



Section 6.1, Problem 10



Section 6.1, Problem 12