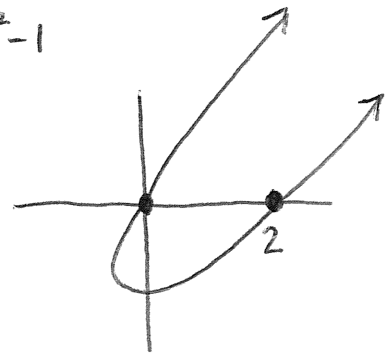
MATH 3503 Homework 1

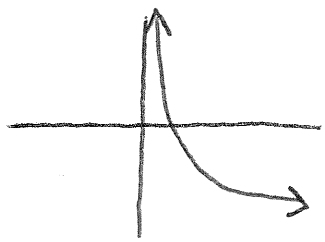
#6 Sketch $\begin{cases} x(t) = t^2+t \\ y(t) = t^2-1 \end{cases}$

Soln: From computer:



#7 Sketch $\begin{cases} x(t) = e^{-t} \\ y(t) = e^{2t}-1 \end{cases}$

Soln: From computer:



#50 Using parametric equation for circle of radius r centered at (h,k):

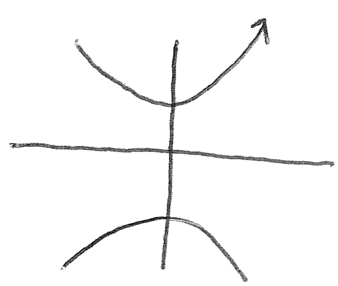
$$\begin{cases} x(t) = h + r\cos(\theta) \\ y(t) = k + r\sin(\theta), \end{cases}$$

find parametric equations for a circle of radius 5 centered at $(\underset{h}{-2}, \underset{k}{3})$

Soln:

$$\begin{cases} x(t) = -2+5\cos(\theta) \\ y(t) = 3+5\sin(\theta) \end{cases}$$

#57 Sketch $\begin{cases} x(t) = 2\tan(t) \\ y(t) = 3\sec(t) \\ -\pi \le t \le \pi \end{cases}$



#66 Find slope of tangent line to $\begin{cases} x(t) = 3\sin(t) \\ y(t) = 3\cos(t) \end{cases}$ at $t = \frac{\pi}{4}$

Soln: Using theorem 1.1: $\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{-3\sin(t)}{3\cos(t)} = -\frac{\sin(t)}{\cos(t)}$, so $\left.\frac{dy}{dx}\right|_{t=\pi/4} = -\frac{\sin(\frac{\pi}{4})}{\cos(\pi/4)} = -1$

Therefore we need tangent line thru point $(x(\frac{\pi}{4}), y(\frac{\pi}{4})) = (\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2})$ with slope -1:

$$y - \frac{3\sqrt{2}}{2} = (-1)\left(x - \frac{3\sqrt{2}}{2}\right)$$

#102 | Find area enclosed by ellipse $\begin{cases} x = a\cos(\theta) \\ y = b\sin(\theta) \end{cases}$ $0 \le \theta \le 2\pi$

Soln: Using Theorem 1.2, (using absolute value to prevent negatives)

$$\text{Area} = \left| \int_0^{2\pi} y(\theta)\, x'(\theta)\, d\theta \right| = \left| \int_0^{2\pi} b\sin(\theta)\left(a\cos(\theta)\right)' d\theta \right|$$

$$= \left| -ab \int_0^{2\pi} \sin^2(\theta)\, d\theta \right|$$

$\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$ $\longrightarrow$

$$= |ab| \left| \int_0^{2\pi} \frac{1-\cos(2\theta)}{2}\, d\theta \right|$$

$$= |ab| \left| \left[\frac{\theta}{2} - \frac{\sin(2\theta)}{4} \right]\Big|_0^{2\pi} \right|$$

$$= |ab| \left| \left[\left(\underbrace{\frac{2\pi}{2}}_{=\pi} - \underbrace{\frac{\sin(4\pi)}{4}}_{=0} \right) - \underbrace{\left(0 - \frac{\sin(0)}{4} \right)}_{=0} \right] \right|$$

$$= \pi |ab|. \blacksquare$$

#116 | Find length of one arch of the cycloid $\begin{cases} x = \theta - \sin(\theta) \\ y = 1 - \cos(\theta) \end{cases}$
(One arch of cycloid occurs for parameter values $0 \le \theta \le 2\pi$)

Soln: Using Theorem 1.3,

$$\text{Arclength} = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{\underbrace{(1-\cos(\theta))^2}_{=1-2\cos(\theta)+\cos^2(\theta)} + (\sin(\theta))^2}\, d\theta$$

$\boxed{\cos^2(\theta)+\sin^2(\theta)=1}$ $\longrightarrow$

$$= \int_0^{2\pi} \sqrt{2 - 2\cos(\theta)}\, d\theta = \sqrt{2} \int_0^{2\pi} \sqrt{1-\cos(\theta)}\, d\theta$$

$\boxed{\begin{array}{c} \cos(2x) = 1 - 2\sin^2(x) \\ \Downarrow \\ 1 - \cos(x) = 2\sin^2\left(\frac{x}{2}\right) \end{array}}$ $\longrightarrow$

$$= \sqrt{2} \int_0^{2\pi} \sqrt{2\sin^2\left(\tfrac{\theta}{2}\right)}\, d\theta$$

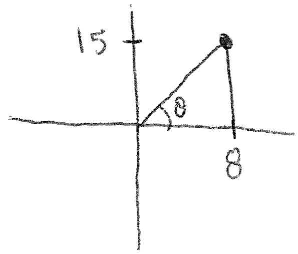
$$= 2\int_0^{2\pi} \sin\left(\tfrac{\theta}{2}\right) d\theta = -4\cos\left(\tfrac{\theta}{2}\right)\Big|_0^{2\pi} = -4\left[\underbrace{\cos(\pi)}_{=-1} - \underbrace{\cos(0)}_{=1}\right]$$

$$= 8$$

§1.3

#138 Find 2 sets of polar coordinates for the rectangular point $(8,15)$.

Soln:



$$r=\sqrt{x^2+y^2}=\sqrt{8^2+15^2}=17$$

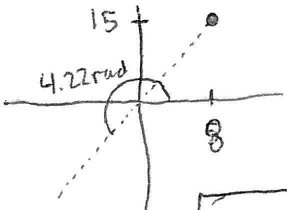
$$\theta=\arctan\left(\frac{15}{8}\right)=1.08 \text{ rad}$$

$$\Downarrow$$

$$\boxed{(r,\theta)=(17,1.08)}$$

Could also think about the angle

$$\theta=\arctan\left(\frac{15}{8}\right)+\pi\approx 4.22 \text{ rad}$$



and take r negative:

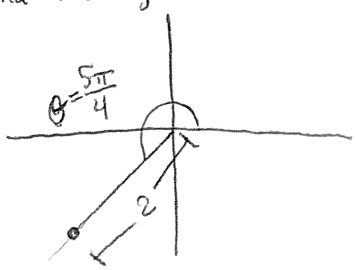
$$r=-17$$

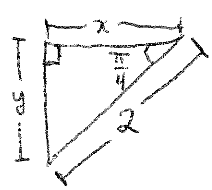
$$\Downarrow$$

$$\boxed{(r,\theta)=(-17,4.22)}$$

Find rectangular coordinates of the polar point $(r,\theta)=\left(2,\frac{5\pi}{4}\right)$

#142





$$\underbrace{\frac{\sqrt{2}}{2}}_{\text{Calculated}}=\sin\left(\frac{\pi}{4}\right)=\underbrace{\frac{y}{2}}_{\text{from }\Delta}$$

AND

$$\underbrace{\frac{\sqrt{2}}{2}}_{\text{Calculated}}=\cos\left(\frac{\pi}{4}\right)=\underbrace{\frac{x}{2}}_{\text{from }\Delta}$$

$$\Downarrow$$

$$\boxed{y=-\sqrt{2}}$$

$$\Downarrow$$

$$\boxed{x=-\sqrt{2}}$$

must be negative b/c point is in QIII

Thus the coordinates are $(x,y)=(-\sqrt{2},-\sqrt{2})$.