MATH 3503 - EXAM 4 FALL 2018 SOLUTION

Friday, 16 November 2018
Instructor: Tom Cuchta

Instructions:

- Show all work, clearly and in order, if you want to get full credit. If you claim something is true **you must show work backing up your claim**. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers algebraically whenever possible to ensure full credit.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point.
- Good luck!

Formulas

Polar coordinates: $\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \\ \mathrm{d}A \mapsto r\mathrm{d}r\mathrm{d}\theta \end{cases}$ **Cylindrical coordinates**: $\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \\ z = z \\ \mathrm{d}V \mapsto r\mathrm{d}r\mathrm{d}\theta \end{cases}$

Spherical coordinates: $\begin{cases} x = \rho\sin(\phi)\cos(\theta) \\ y = \rho\sin(\phi)\sin(\theta) \\ z = \rho\cos(\phi) \\ \mathrm{d}V \mapsto \rho^2\sin(\phi)\mathrm{d}\rho\mathrm{d}\theta\mathrm{d}\phi \end{cases}$

Parametrizations

From point P to point Q: $\begin{cases} \vec{r}(t) = tQ + (1-t)P \\ 0 \leq t \leq 1 \end{cases}$

Of the curve $y = f(x)$ between x values $x = a$ and $x = b$: $\begin{cases} \vec{r}(t) = \langle t, f(t) \rangle \\ a \leq t \leq b \end{cases}$

Let C be a curve parametrized by $\vec{r}(t)$ for $a \leq t \leq b$:

Scalar line integral ("magic formula"): $\displaystyle\oint_C f\mathrm{d}s = \int_a^b f(\vec{r}(t))\|\vec{r}'(t)\|\mathrm{d}t$

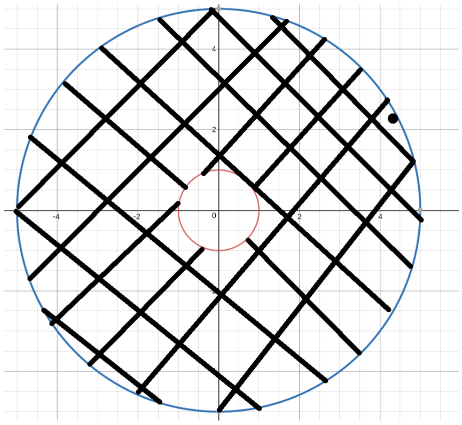
Vector field line integral ("magic formula"): $\displaystyle\oint_C \vec{F}\cdot\mathrm{d}\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}'(t)\mathrm{d}t$

Green's theorem: if C surrounds the region D, then

$$\oint_C \vec{F}\cdot\mathrm{d}\vec{r} = \iint_D \frac{\partial}{\partial x} - \frac{\partial P}{\partial y}\mathrm{d}A$$

1. (12 points)

   (a) (5 points) Draw the region D in the plane bounded by the circles $x^2+y^2=1$ and $x^2+y^2=25$.
   Solution:



   (b) (7 points) Compute $\iint_D 3-x^2-y^2 \mathrm{d}A$, where D is the region in part (a).

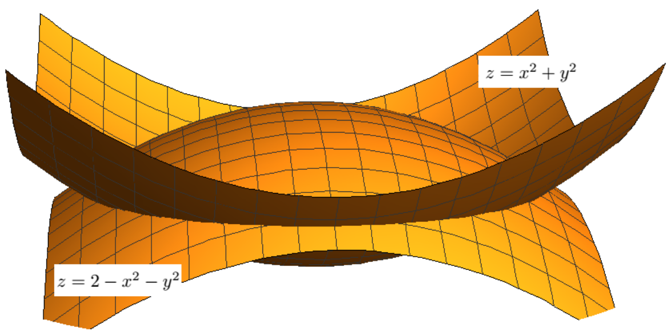
   Solution: This region can be described in polar coordinates as

$$D=\{(r,\theta)\colon 1\le r\le 5, 0\le\theta\le 2\pi\}.$$

   In polar coordinates, $3-x^2-y^2=3-(x^2+y^2)=3-r^2$. Therefore we have

$$\begin{aligned}\iint_D 3-x^2-y^2\mathrm{d}A &= \int_0^{2\pi}\int_1^5 (3-r^2)r\mathrm{d}r\mathrm{d}\theta\\ &= \int_0^{2\pi} \frac{3r^2}{2}-\frac{r^4}{4}\Big|_{r=1}^{r=5}\mathrm{d}\theta\\ &= \left(\left(\frac{75}{2}-\frac{625}{4}\right)-\left(\frac{3}{2}-\frac{1}{4}\right)\right)\int_0^{2\pi}1\mathrm{d}\theta\\ &= -120(2pi)\\ &= -240\pi\end{aligned}.$$

2. (14 points) Consider the region E between the surfaces $z=x^2+y^2$ and $z=2-x^2-y^2$.



(a) (4 points) Find a formula for the intersection curve of these two surfaces (*hint: each are "$z =$", so set them equal!*).
Solution: We have

$$x^2 + y^2 = z = 2 - x^2 - y^2.$$

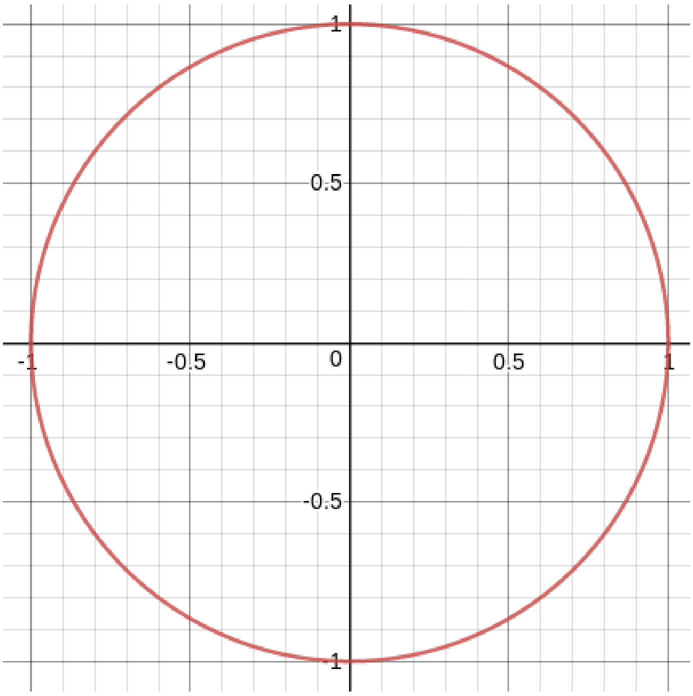
Therefore

$$2(x^2 + y^2) = 2,$$

hence

$$x^2 + y^2 = 1.$$

(b) (4 points) Draw a picture of the curve you found in (a) in the xy-plane.
Solution: The curve is the circle of radius 1 centered at $(0, 0)$:



(c) (6 points) Set up **but do not evaluate** the triple interal $\iiint_E x + y + z\mathrm{d}V$ **in cylindrical coordinates**.
Solution: We convert the surfaces in the problem into cylindrical coordinates: the lower surface is $z = x^2 + y^2 = r^2$ and the upper surface is $z = 2 - x^2 - y^2 = 2 - r^2$. We "shoot the arrow" in the z direction and compute

$$\iiint_E x + y + z\mathrm{d}V = \int_0^{2\pi} \int_0^1 \int_{r^2}^{2-r^2} \Big(r\cos(\theta) + r\sin(\theta) + z\Big) r\mathrm{d}z\mathrm{d}r\mathrm{d}\theta$$

3. (12 points) Set up **but do not evaluate** the following triple integral as a triple integral in spherical coordinates:

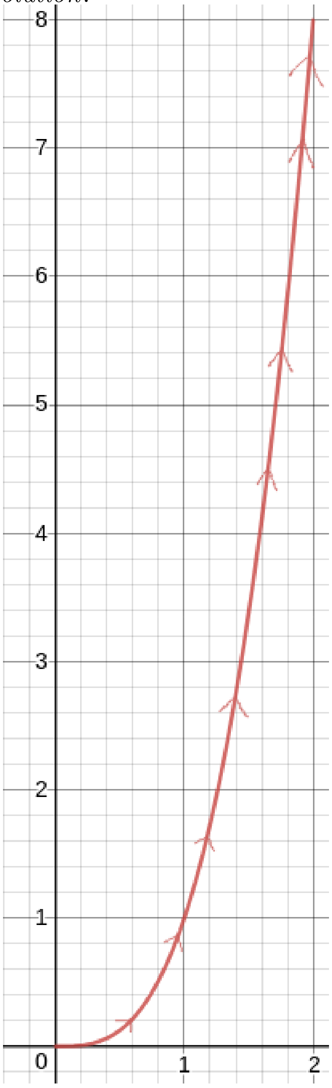
$$\int_0^3 \int_{-\sqrt{9-y^2}}^0 \int_0^{\sqrt{9-x^2-y^2}} z \ \mathrm{d}z\mathrm{d}x\mathrm{d}y$$

Solution: The surfaces: $z = 0$ is the xy-plane and $z = \sqrt{9 - x^2 - y^2}$ is the upper half of the sphere of radius 3 centered at $(0, 0, 0)$. The shadow region is bounded by the surfaces $x = -\sqrt{9 - y^2}$ which is the left half of the semicircle of radius 3 centered at $(0, 0)$ and $x = 0$, which is the y-axis. The bounds $y = 0$ and $y = 3$ restrict attention to quadrant II. The sphere implies $0 \le \rho \le 3$, the quadrant implies $\frac{\pi}{2} \le \theta \le \pi$, and since only the upper half of the sphere is being considered, $0 \le \phi \le \frac{\pi}{2}$. Note that in spherical coordinates, $z = \rho\cos(\phi)$. Therefore, compute

$$\int_0^3 \int_{-\sqrt{9-y^2}}^0 \int_0^{\sqrt{9-x^2-y^2}} z \ \mathrm{d}z\mathrm{d}x\mathrm{d}y = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^3 (\rho\cos(\phi))\rho^2 \sin(\phi)\mathrm{d}\rho\mathrm{d}\theta\mathrm{d}\phi.$$

4. (12 points) (a) (6 points) **Draw** and parametrize the curve C defined as the part of the curve $y = x^3$ from $(0, 0)$ to $(2, 8)$.

Solution:



We parametrize the curve as a function of the form $y = f(x)$ as

$$\begin{cases} \vec{r}(t) = \left\langle t, t^3 \right\rangle \\ 0 \le t \le 2 \end{cases}$$

(b) (6 points) Use your answer in (a) to set up **but do not evaluate** the line integral $\oint_C x^2 + y\mathrm{d}s$.

Solution: This is a scalar line integral, so we first compute $\vec{r}\,'(t) = \left\langle 1, 3t^2 \right\rangle$ and hence we see $\|\vec{r}\,'(t)\| = \sqrt{1+9t^4}$. Now we can compute

$$\oint_C x^2 + y\mathrm{d}s \quad = \int_0^2 (t^2+t^3)\sqrt{1+9t^4}\mathrm{d}t$$

5. (12 points) (a) (5 points) Parametrize the curve C defined as the line segement from $(1,0,0)$ to $(0,1,1)$.
Solution: We parametrize this line segment as

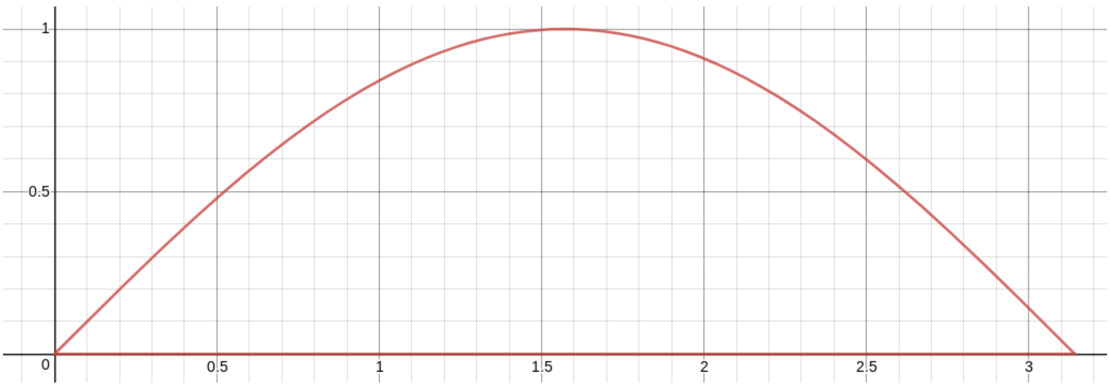
$$\begin{cases} \vec{r}(t) = t\langle 0,1,1\rangle + (1-t)\langle 1,0,0\rangle = \langle 1-t, t, t\rangle \\ 0 \le t \le 1. \end{cases}$$

(b) (7 points) Set up **but do not evaluate** the line integral $\oint_C \langle z, -x, y\rangle \cdot \mathrm{d}\vec{r}$.
Solution: This is a vector field line integral. So compute $\vec{r}\,'(t) = \langle -1, 1, 1\rangle$. Therefore compute

$$\oint_C \langle z, -x, y\rangle \cdot \mathrm{d}\vec{r} = \int_0^1 \langle t, t-1, t\rangle \cdot \langle -1,1,1\rangle \mathrm{d}t = \int_0^1 -t + (t-1) + t\mathrm{d}t = \int_0^1 t - 1\mathrm{d}t$$

6. (13 points) Consider the contour C (oriented counterclockwise) which starts at $(0,0)$, travels along the x-axis until it reaches $(\pi, 0)$ and then goes back to $(0,0)$ along the curve $y = \sin(x)$.

(a) (5 points) Draw this curve.
Solution: Draw:



(b) (8 points) Use your drawing from (a) and Green's theorem to compute

$$\oint_C \langle x^2 e^x - y, x - \ln(y)e^y\rangle \cdot \mathrm{d}\vec{r}.$$

Solution: In this case $P = x^2e^x - y$ and $Q = x - \ln(y)e^y$. We may compute $P_y = -1$ and $Q_x = 1$. The region in question is bounded between the curves $y = 0$ and $y = \sin(x)$ between

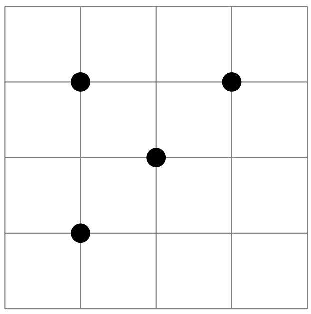
x-values $x = 0$ and $x = \pi$. Therefore by Green's theorem,

$$\oint_C \langle x^2 e^x - y, x - \ln(y)e^y \rangle \cdot \mathrm{d}\vec{r} \overset{\text{Green's theorem}}{=} \int_0^{\pi} \int_0^{\sin(x)} \underbrace{Q_x - P_y}_{=1-(-1)=2} \,\mathrm{d}y\mathrm{d}x$$
$$= 2\int_0^{\pi} \sin(x)\mathrm{d}x$$
$$= -2\cos(x)\Big|_0^{\pi}$$
$$= -2\cos(\pi) - (-2\cos(0))$$
$$= (-2)(-1) + 2(1)$$
$$= 4.$$

7. (8 points) Consider the function $f(x, y, z) = x^2 \sin(yz) - ye^{\cos(x)}$. Use the fundamental theorem of line integrals to calculate $\oint_C \nabla f \cdot \mathrm{d}\vec{r}$ when C is any path from $(0, 0, 0)$ to $(0, 2, 5)$.
Solution: By the fundamental theorem of line integrals,

$$\oint_C \nabla f \cdot \mathrm{d}\vec{r} = f(0, 2, 5) - f(0, 0, 0) = (0 - 2e^{\cos(0)}) - (0 - 0) = -2.$$

8. (8 points) Draw vectors at each of the four following dots determined by the vector field $\vec{F} = \langle -x + 1, -x - y \rangle$ (the center point is at the point $(0, 0)$).



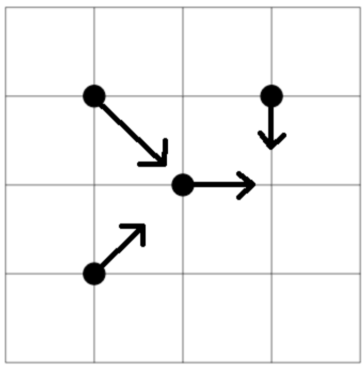
Solution: First compute

$$\vec{F}(-1, 1) = \langle -(-1) + 1, -(-1) - 1 \rangle = \langle 2, -2 \rangle,$$
$$\vec{F}(-1, -1) = \langle -(-1) + 1, -(-1) - (-1) \rangle = \langle 2, 2 \rangle,$$
$$\vec{F}(0, 0) = \langle 0 + 1, 0 \rangle = \langle 1, 0 \rangle,$$

and

$$\vec{F}(1, 1) = \langle -1 + 1, -1 - 1 \rangle = \langle 0, -2 \rangle.$$

Drawing these yields



9. (9 points) Consider the vector field $\vec{F} = \langle x^2y, 3x - z^3, 4y^2 \rangle$.

(a) (5 points) Compute curl $\vec{F}$. *Solution*: Compute

$$\begin{aligned}
\text{curl}\langle x^2y, 3x - z^3, 4y^2 \rangle &= \nabla \times \langle x^2y, 3x - z^3, 4y^2 \rangle \\
&= \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2y & 3x - z^3 & 4y^2 \end{bmatrix} \\
&= \langle 8y + 3z^2, 0, 3 - x^2 \rangle
\end{aligned}$$

(b) (4 points) Compute div $\vec{F}$.
Solution: Compute

$$\begin{aligned}
\text{div}\langle x^2y, 3x - z^3, 4y^2 \rangle &= \nabla \cdot \langle x^2y, 3x - z^3, 4y^2 \rangle \\
&= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle x^2y, 3x - z^3, 4y^2 \rangle \\
&= \frac{\partial}{\partial x}\left[x^2y\right] + \frac{\partial}{\partial y}\left[3x - z^3\right] + \frac{\partial}{\partial z}\left[4y^2\right] \\
&= 2xy + 0 + 0 \\
&= 2xy.
\end{aligned}$$