MATH 3503 - EXAM 2 - FALL 2018
SOLUTION

4 October 2018
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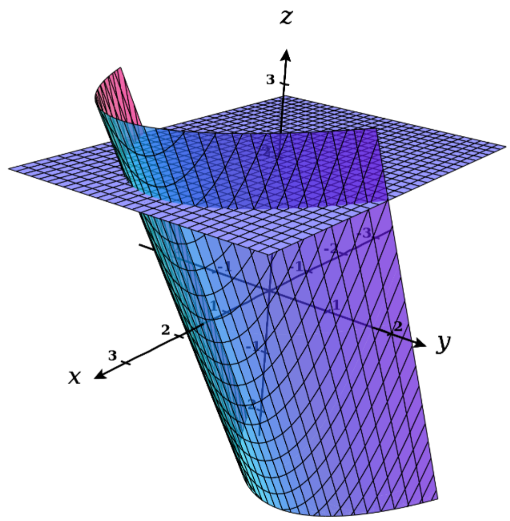
Instructions:

- Show all work, clearly and in order, if you want to get full credit. If you claim something is true **you must show work backing up your claim**. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers algebraically whenever possible to ensure full credit.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point.
- Good luck!

1. (a) Find an equation of the form "$x = ...$" for the level curve at height 2 of the surface $z = 2x + y^2 - 3$.



Solution: The level surface is given by setting $z = 2$ in the equation for the surface. This gives us

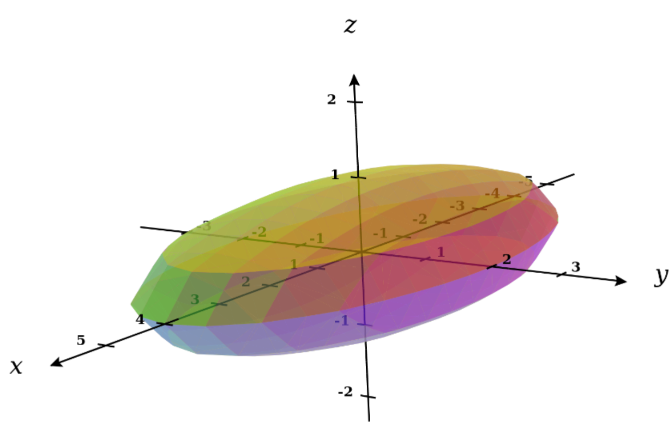
$$2 = 2x + y^2 - 3.$$

Therefore solving for x yields

$$x = \frac{2 - y^2 + 3}{2}.$$

(b) Find and classify the x-traces of the ellipsoid

$$\frac{x^2}{16} + \frac{y^2}{4} + z^2 = 1.$$



Solution: The x-traces are given by setting $x = k$ for various values of k. Doing this gives the equation

$$\frac{y^2}{4} + z^2 = 1 - \frac{k^2}{16}.$$

$\underline{k < -4 \text{ or } k > 4}$	$\underline{k = 4, k = -4}$	$\underline{-4 < k < 4}$
In this case, the right-hand side is negative, but it is impossible for the sum on the left to add up to a negative number (assuming y and z are real numbers, which we are!!), so this is **empty**.	In this case, the right-hand side is exactly zero, meaning the only solutions are the single **point** $(y, z) = (0, 0)$.	In this case, the right-hand side is a number between 0 and 1. This means that the equations define an **ellipse** in the yz-plane.

2. Let acceleration of a particle be given by $\vec{a}(t) = \langle t, 1, 1\rangle$. Assume at time $t = 0$, the particle had initial velocity $\vec{v}(0) = \langle 1, 0, 1\rangle$ and assume that at time $t = 1$, the particle was at at $\vec{r}(1) = \langle 1, 1, 0\rangle$.

   (a) Find the velocity function $\vec{v}(t)$ for all times t if $\vec{v}(0) = \langle 1, 0, 1\rangle$.
   Solution: Integrate to get

$$\vec{v}(t) = \int \vec{a}(t)\mathrm{d}t = \left\langle \frac{t^2}{2}, t, t \right\rangle + \vec{c},$$

   where $\vec{c}$ is an unknown vector. To find $\vec{c}$, consider

$$\underbrace{\langle 1, 0, 1\rangle}_{\text{given}} = \vec{v}(0) = \underbrace{\langle 0, 0, 0\rangle + \vec{c}}_{\text{computed}}.$$

   Therefore $\vec{c} = \langle 1, 0, 1\rangle$. So we know that

$$\vec{v}(t) = \left\langle \frac{t^2}{2} + 1, t, t + 1 \right\rangle.$$

   (b) Using your answer from (a), find the position function $\vec{r}(t)$ for all times t if $\vec{r}(1) = \langle 1, 1, 0\rangle$.
   Solution: Integrating again yields the position function:

$$\vec{r}(t) = \int \vec{v}(t)\mathrm{d}t = \left\langle \frac{t^3}{6} + t, \frac{t^2}{2}, \frac{t^2}{2} + t \right\rangle + \vec{d},$$

   where $\vec{d}$ is an unknown vector. To find $\vec{d}$, consider

$$\underbrace{\langle 1, 1, 0\rangle}_{\text{given}} = \vec{r}(1) = \underbrace{\left\langle \frac{7}{6}, \frac{1}{2}, \frac{3}{2} \right\rangle + \vec{d}}_{\text{computed}}.$$

   Therefore

$$\vec{d} = \left\langle -\frac{1}{6}, \frac{1}{2}, -\frac{3}{2} \right\rangle.$$

   So we know that

$$\vec{r}(t) = \left\langle \frac{t^3}{6} + t - \frac{1}{6}, \frac{t^2}{2} + \frac{1}{2}, \frac{t^2}{2} + t - \frac{3}{2} \right\rangle.$$

3. Consider the vector-valued function $\vec{r}(t) = \langle \cos(t), t, \sin(t)\rangle$.

   (a) Find the unit tangent vector $\vec{T}(t)$.
   Solution: First compute

$$\vec{r}\,'(t) = \langle -\sin(t), 1, \cos(t)\rangle,$$

   and

$$\|\vec{r}\,'\| = \sqrt{(-\sin(t))^2 + 1^2 + \cos^2(t)} = \sqrt{2}$$

   So, by definition,

$$\vec{T} = \frac{\vec{r}\,'}{\|\vec{r}\,'\|} = \left\langle \frac{-\sin(t)}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{\cos(t)}{\sqrt{2}} \right\rangle$$

(b) Use your answer from (b) to compute the unit normal vector $\vec{N}(t)$.
Solution: Compute

$$\vec{T}\,' = \left\langle \frac{-\cos(t)}{\sqrt{2}}, 0, \frac{-\sin(t)}{\sqrt{2}} \right\rangle,$$

hence

$$\left\|\vec{T}\,'\right\| = \sqrt{\frac{(-\cos(t))^2}{2} + \frac{(-\sin(t))^2}{2}} = \frac{1}{\sqrt{2}}.$$

Therefore, by definition,

$$\vec{N}\,' = \frac{\vec{T}\,'}{\left\|\vec{T}\,'\right\|} = \frac{\left\langle \frac{-\cos(t)}{\sqrt{2}}, 0, \frac{-\sin(t)}{\sqrt{2}} \right\rangle}{\frac{1}{\sqrt{2}}} = \langle -\cos(t), 0, -\sin(t) \rangle\,.$$

4. Ohm's law says that the current through a conductor between two points is directly proportional to the voltage across the two points and obeys the equation

$$I = \frac{V}{R},$$

where I is the current through the conductor (in amps) , V is the voltage (in volts), and R is the resistance (in ohms).

(a) Find $\frac{\partial I}{\partial V}$ and explain what this quantity represent.
Solution:

$$\frac{\partial I}{\partial V} = \frac{1}{R}\frac{\text{amps}}{\text{volt}}$$

(b) Find $\frac{\partial I}{\partial R}$ and explain what this quantity represent.
Solution:

$$\frac{\partial I}{\partial R} = -\frac{V}{R^2}\frac{\text{amps}}{\text{ohm}}.$$

5. Consider $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{2x^2+y^2}$.

(a) Compute the limit on a path along the x-axis.
Solution: This path looks like $(x, 0)$ as $x \to 0$. Therefore

$$\lim_{(x,y)\to(0,0)} \frac{xy}{2x^2+y^2} = \lim_{(x,0)\to(0,0)} 0 = \lim_{x\to 0} 0 = 0$$

(b) Compute the limit on a path along the line $y = x$.
Solution: This path looks like (x, x) as $x \to 0$. Therefore,

$$\lim_{(x,y)\to(0,0)} \frac{xy}{2x^2+y^2} = \lim_{(x,x)\to(0,0} \frac{x^2}{3x^2} = \lim_{x\to 0} \frac{1}{3} = \frac{1}{3}.$$

(c) From your results above, what can you say about the limit in question?
Solution: Since the limits disagree, we can conclude that the limit in question does not exist, since it approaches different values along different paths.

6. Consider the function $f(x, y) = ye^{2x} + y^2\cos(\pi x)$.

(a) Compute $\dfrac{\partial f}{\partial x}$.
Solution: Compute

$$\frac{\partial f}{\partial x} = 2ye^{2x} - \pi y^2 \sin(\pi x)$$

(b) Compute $\dfrac{\partial f}{\partial y}$.
Solution: Compute

$$\frac{\partial f}{\partial y} = e^{2x} + 2y\cos(\pi x)$$

(c) Using your answers to (a) and (b), find an equation of the tangent plane of f at the point $(0, 1, 2)$
Solution: Compute

$$\left.\frac{\partial f}{\partial x}\right|_{(x,y)=(0,1)} = 2$$

and

$$\left.\frac{\partial f}{\partial y}\right|_{(x,y)=(0,1)} = 1 + 2 = 3$$