

# MATH 3503 - EXAM 1 - FALL 2018

## SOLUTION

7 September 2018  
Instructor: Tom Cuchta

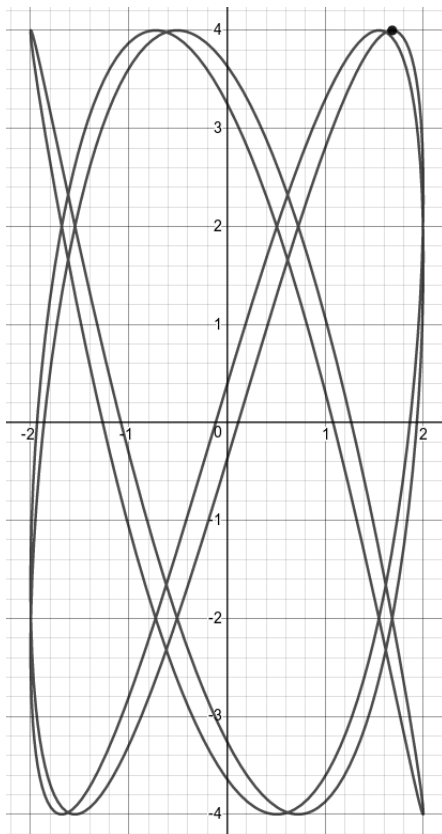
### Instructions:

- Show all work, clearly and in order, if you want to get full credit. If you claim something is true **you must show work backing up your claim**. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers algebraically whenever possible to ensure full credit.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point.
- Good luck!

1. (7 points) The curve given by the following parametric equation is called a Lissajous curve:

$$\begin{cases} x = 2 \sin(3t + 1) \\ y = 4 \cos(5t) \end{cases}$$

We may draw it; the point corresponding to  $t = 0$  is indicated with a filled black circle at the top right of the plot:



Find an **equation** for the tangent line to this curve at the point corresponding to  $t = 0$ .

*Solution (this problem resembles #66 from HW1):* First find the point corresponding to parameter value  $t = 0$  by plugging  $t = 0$  into the parametric equations:

$$x(0) = 2 \sin(3(0) + 1) = 2 \sin(1),$$

and

$$y(0) = 4 \cos(5(0)) = 4 \underbrace{\cos(0)}_{=1} = 4.$$

Thus we see that the coordinates of the point in question are  $(x_0, y_0) = (2 \sin(1), 4)$ . Calculate

$$x'(t) = 6(1 - \cos(t)),$$

and

$$y'(t) = 6 \sin(t).$$

Now calculate the slope of the tangent line at any parameter value  $t$ :

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{6 \sin(t)}{6(1 - \cos(t))}.$$

Therefore the slope of the tangent line at the point of the graph corresponding to parameter  $t = 0$  is

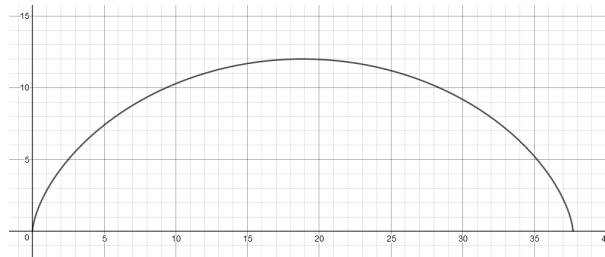
$$\left. \frac{dy}{dx} \right|_{t=0} = \frac{6 \sin(0)}{6(1 - \cos(0))} = \frac{0}{6} = 0.$$

Therefore using the point-slope form of the equation of the line, we get the equation to the tangent line to be the line going through  $(x_0, y_0) = (2 \sin(1), 4)$  with slope  $m = 0$ , i.e.

$$y - 4 = 0,$$

which is the horizontal line  $y = 4$ .

2. (8 points) Find the area under the cycloid given by
- $$\begin{cases} x = 6(\theta - \sin(\theta)) \\ y = 6(1 - \cos(\theta)) \\ 0 \leq \theta \leq 2\pi \end{cases}$$



*Solution (this problem resembles #102 in HW1):* First calculate

$$x'(\theta) = 6(1 - \cos(\theta)).$$

Now, using the formula for area, we compute

$$\begin{aligned} (*) \quad \text{Area} &= \left| \int_0^{2\pi} y(t)x'(t)d\theta \right| \\ &= \left| \int_0^{2\pi} 6(1 - \cos(\theta))6(1 - \cos(\theta))d\theta \right| \\ &= 36 \left| \int_0^{2\pi} 1 - 2\cos(\theta) + \cos^2(\theta)d\theta \right| \\ &= 36 \left[ \left| \int_0^{2\pi} 1d\theta - 2 \int_0^{2\pi} \cos(\theta)d\theta + \int_0^{2\pi} \cos^2(\theta)d\theta \right| \right] \end{aligned}$$

Since

$$\int_0^{2\pi} 1 d\theta = \theta \Big|_0^{2\pi} = 2\pi - 0 = 2\pi,$$

$$-2 \int_0^{2\pi} \cos(\theta) d\theta = -2 \sin(\theta) \Big|_0^{2\pi} = -2 \sin(2\pi) - (-2) \sin(0) = 0 - 0 = 0,$$

and

$$\begin{aligned} \int_0^{2\pi} \cos^2(\theta) d\theta &\stackrel{\text{trig identity}}{=} \int_0^{2\pi} \frac{1 + \cos(2\theta)}{2} d\theta \\ &= \int_0^{2\pi} \frac{1}{2} d\theta + \frac{1}{2} \int_0^{2\pi} \cos(2\theta) d\theta \\ &= \frac{1}{2}(2\pi - 0) + \frac{1}{2} \left[ \frac{\sin(2\theta)}{2} \right]_0^{2\pi} \\ &= \pi + 0 \\ &= \pi, \end{aligned}$$

we now see from (\*) that

$$\text{Area} = 36[2\pi - 0 + \pi] = 36(3\pi) = 108\pi.$$

3. (8 points) Find the arc length of the parametric curve

$$\left\{ \begin{array}{l} x = \cos(t) \\ y = \sin(t) \\ z = \frac{t}{2} \\ 1 \leq t \leq 5 \end{array} \right.$$

*Solution* (this problem resembles #116 in HW1): First calculate

$$x'(t) = -\sin(t),$$

$$y'(t) = \cos(t),$$

and

$$z'(t) = \frac{1}{2}.$$

Now using the arclength formula, we calculate

$$\begin{aligned}\text{ArcLength} &= \int_1^5 \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} dt \\ &= \int_1^5 \sqrt{(-\sin(t))^2 + (\cos(t))^2 + \left(\frac{1}{2}\right)^2} dt \\ &\stackrel{\cos^2(t) + \sin^2(t) = 1}{=} \int_1^5 \sqrt{1 + \frac{1}{4}} dt \\ &= \sqrt{\frac{5}{4}} \int_1^5 1 dt \\ &= \sqrt{\frac{5}{4}} \Big|_{t=1}^{t=5} \\ &= \sqrt{\frac{5}{4}} (5 - 1) \\ &= 4\sqrt{\frac{5}{4}} \\ &= 2\sqrt{5}.\end{aligned}$$

4. (18 points) Consider the vectors  $\vec{a} = \langle 1, 4, -1 \rangle$  and  $\vec{b} = \langle -1, -2, 3 \rangle$ .

(a) (6 points) Compute  $2\vec{a} - 3\vec{b}$ .

*Solution (this resembles Quiz 1 (i)):* Calculate

$$\begin{aligned}2\vec{a} - 3\vec{b} &= 2\langle 1, 4, -1 \rangle - 3\langle -1, -2, 3 \rangle \\ &= \langle 2, 8, -2 \rangle - \langle -3, -6, 9 \rangle \\ &= \langle 2 + 3, 8 + 6, -2 - 9 \rangle \\ &= \langle 5, 14, -11 \rangle.\end{aligned}$$

(b) (6 points) Compute  $\|\vec{a} + \vec{b}\|$ .

*Solution (this resembles Quiz 1 (ii)):* Calculate

$$\begin{aligned}\|\vec{a} + \vec{b}\| &= \|\langle 1, 4, -1 \rangle + \langle -1, -2, 3 \rangle\| \\ &= \|\langle 1 - 1, 4 - 2, -1 + 3 \rangle\| \\ &= \|\langle 0, 2, 2 \rangle\| \\ &= \sqrt{0^2 + 2^2 + 2^2} \\ &= \sqrt{8}.\end{aligned}$$

(c) (6 points) Find the unit vector  $\vec{u}$  pointing in the same direction as  $\vec{a}$ .

*Solution (this resembles #10 in HW2):* We know that  $\vec{a}$  is not a unit vector because

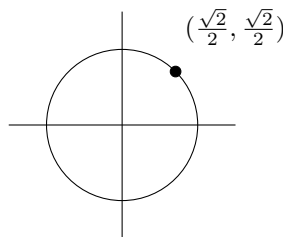
$$\|\vec{a}\| = \|\langle 1, 4, -1 \rangle\| = \sqrt{1^2 + 4^2 + (-1)^2} = \sqrt{1 + 16 + 1} = \sqrt{18} \underbrace{\neq 1}_{\text{not a unit vector}}.$$

Therefore we must scale  $\vec{a}$  to get a unit vector pointing in the same direction as  $\vec{a}$  as follows:

$$\vec{u} = \frac{\vec{a}}{\|\vec{a}\|} = \frac{1}{\sqrt{18}} \langle 1, 4, -1 \rangle = \left\langle \frac{1}{\sqrt{18}}, \frac{4}{\sqrt{18}}, -\frac{1}{\sqrt{18}} \right\rangle.$$

5. (7 points) Find the unit vector  $\vec{v}$  in the plane with magnitude  $\|\vec{v}\| = 25$  that makes an angle of  $\frac{\pi}{4}$  radians with the positive  $x$ -axis.

*Solution (this resembles #34 in HW2):* Consider the following portion of the unit circle:



So we observe that a unit vector (it is length 1 because it comes from a point on the **unit** circle) in the desired direction is  $\vec{u} = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$ . Therefore the desired vector is

$$\vec{v} = 25\vec{u} = 25 \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle = \langle 25\sqrt{2}, 25\sqrt{2} \rangle.$$

6. (14 points) Let  $\vec{u} = \langle 1, 3, -1 \rangle$  and let  $\vec{v} = \langle 5, -1, -2 \rangle$ .

(a) (7 points) Compute  $\text{proj}_{\vec{u}}\vec{v}$ .

*Solution (this problem resembles #170 a) in HW2):* Using the definition of  $\text{proj}$ , compute

$$\begin{aligned}\text{proj}_{\vec{u}}\vec{v} &= \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|^2} \vec{u} \\ &= \frac{\langle 1, 3, -1 \rangle \cdot \langle 5, -1, -2 \rangle}{\|\langle 1, 3, -1 \rangle\|^2} \langle 1, 3, -1 \rangle \\ &= \frac{5 - 3 + 2}{\sqrt{1 + 9 + 1}^2} \langle 1, 3, -1 \rangle \\ &= \frac{4}{11} \langle 1, 3, -1 \rangle \\ &= \left\langle \frac{4}{11}, \frac{12}{11}, -\frac{4}{11} \right\rangle.\end{aligned}$$

(b) (7 points) Use your answer from above to compute  $\text{comp}_{\vec{u}}\vec{v}$ .

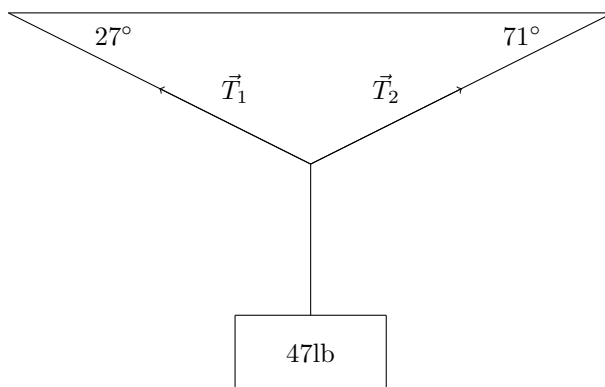
*Solution (this problem resembles #170 b) in HW2):* Using the definition of  $\text{comp}$ , compute

$$\text{comp}_{\vec{u}}\vec{v} = \|\text{proj}_{\vec{u}}\vec{v}\| = \left\| \left\langle \frac{4}{\sqrt{11}}, \frac{12}{\sqrt{11}}, -\frac{4}{\sqrt{11}} \right\rangle \right\| = \sqrt{\frac{16}{121} + \frac{144}{121} + \frac{16}{121}} = \sqrt{\frac{176}{121}} = \frac{\sqrt{176}}{11}.$$

7. (18 points) A 47 lb weight hangs from a rope that makes the angles of  $27^\circ$  and  $71^\circ$  respectively with the horizontal.

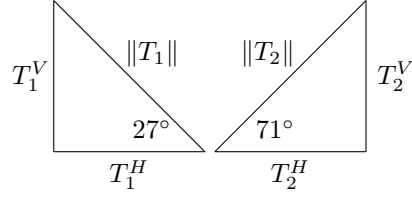
(a) (5 points) Draw a picture of this scenario, letting  $\vec{T}_1$  and  $\vec{T}_2$  be the force of tension in the cables.

*Solution (this resembles # 56 on HW2):* Draw the scenario:



(b) (13 points) Find the **magnitudes** of the forces of tension  $\vec{T}_1$  and  $\vec{T}_2$  in the cables if the resultant force of the object is zero.

*Solution:* First draw the two natural triangles in this problem:



From this we observe:

$$T_1^H = \|T_1\| \cos(27^\circ),$$

$$T_1^V = \|T_1\| \sin(27^\circ),$$

$$T_2^H = \|T_2\| \cos(27^\circ),$$

and

$$T_2^V = \|T_2\| \sin(27^\circ).$$

Since “result force of the object is zero”, we arrive at the equations

$$\begin{cases} T_1^H = T_2^H \\ T_1^V + T_2^V = 47 \end{cases}$$

Replacing these with the equivalent expressions involving  $\|T_1\|$  and  $\|T_2\|$ , we obtain

$$\begin{cases} \|T_1\| \cos(27^\circ) = \|T_2\| \cos(71^\circ) & (i) \\ \|T_1\| \sin(27^\circ) + \|T_2\| \sin(71^\circ) = 47 & (ii) \end{cases}$$

From (i) we get

$$\|T_1\| = \|T_2\| \frac{\cos(71^\circ)}{\cos(27^\circ)}.$$

Plugging this into (ii) yields

$$\left( \|T_2\| \frac{\cos(71^\circ)}{\cos(27^\circ)} \right) \sin(27^\circ) + \|T_2\| \sin(71^\circ) = 47.$$

Solving this for  $\|T_2\|$  yields

$$\|T_2\| = \frac{47}{\frac{\cos(71^\circ)}{\cos(27^\circ)} \sin(27^\circ) + \sin(71^\circ)},$$

and hence

$$\|T_1\| = \left( \frac{47}{\frac{\cos(71^\circ)}{\cos(27^\circ)} \sin(27^\circ) + \sin(71^\circ)} \right) \frac{\cos(71^\circ)}{\cos(27^\circ)}.$$



8. (6 points) Let  $P = (1, 2, 3)$  and let  $Q = (5, -1, 2)$ . Find an equation for the line containing both  $P$  and  $Q$ . *Solution (this problem resembles HW3 #244 and #248):* The parallel vector is given by

$$\vec{v} = \overrightarrow{PQ} = \langle 5, -1, 2 \rangle - \langle 1, 2, 3 \rangle = \langle 4, -3, -1 \rangle.$$

Therefore, using  $(x_0, y_0, z_0) = (1, 2, 3)$  we observe that the equation of the line is given by the following parametric equations:

$$\begin{cases} x = 1 + 4t \\ y = 2 - 3t \\ z = 3 - t \end{cases}$$

9. (14 points) Let  $P = (1, 2, 2)$ , let  $Q = (2, -1, 4)$ , and let  $R = (2, 0, 0)$ . *(this problem resembles HW3 #268)*

- (a) (4 points) Find  $\overrightarrow{PQ}$  and  $\overrightarrow{PR}$ .

*Solution:* Calculate

$$\overrightarrow{PQ} = \langle 2, -1, 4 \rangle - \langle 1, 2, 2 \rangle = \langle 1, -3, 2 \rangle$$

and calculate

$$\overrightarrow{PR} = \langle 2, 0, 0 \rangle - \langle 1, 2, 2 \rangle = \langle 1, -2, -2 \rangle$$

- (b) (5 points) Find a vector  $\vec{n}$  that is perpendicular to both  $\overrightarrow{PQ}$  and  $\overrightarrow{PR}$ .

*Solution:* Calculate

$$\begin{aligned} \vec{n} &= \overrightarrow{PQ} \times \overrightarrow{PR} \\ &= \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -3 & 2 \\ 1 & -2 & -2 \end{pmatrix} \\ &= \langle (-3)(-2) - (2)(-2), -(1(-2) - 2(1)), 1(-2) - (-3)(1) \rangle \\ &= \langle 10, 4, 1 \rangle \end{aligned}$$

- (c) (5 points) Use your answer from part b to find an equation of the plane containing  $P$ ,  $Q$ , and  $R$ .

*Solution:* We write the equation of the plane containing the point  $(x_0, y_0, z_0) = (1, 2, 2)$  with normal vector  $\vec{n} = \langle 10, 4, 1 \rangle$ :

$$10(x - 1) + 4(y - 2) + (z - 2) = 0.$$