Homework 9 — MATH 2510 Spring 2018

Recall that the notation $f\colon A \to B$ is how you say that there is a function named "f" whose domain (i.e. set of inputs) is "A" and whose codomain (i.e. space where outputs live) is "B". Also recall that the range of a function is the set of actual outputs, i.e.

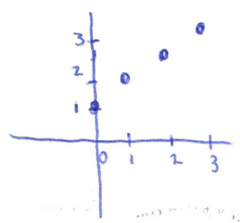
$$\text{range}(f) = \{y \in B \colon \exists x\, f(x) = y\}.$$

We defined a function $f\colon X \to Y$ to be one-to-one provided that $\forall x \forall y (f(x) = f(y) \to x = y)$. We defined the cardinality (i.e. "number of elements") $|A|$ of a set A in the following way: we say that $|X| \leq |Y|$ provided there is a one-to-one function $f\colon X \to Y$. We say that $|X| = |Y|$ (i.e. "the cardinality of X is equal to the cardinality of Y") provided that $|X| \leq |Y|$ and $|Y| \leq |X|$. Of course if a set is finite, we simply report its size as an number; e.g. $|\{a, b, c\}| = 3$.

1. Sketch each function (to the best of your abilities). Is the function one-to-one or not?

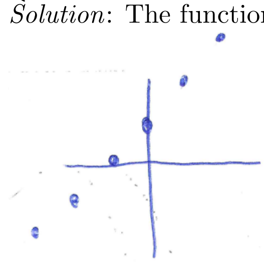
   a.) $\begin{cases} f\colon \mathbb{N} \to \mathbb{N} \\ f(n) = n + 1 \end{cases}$

   Solution: The function is one-to-one.



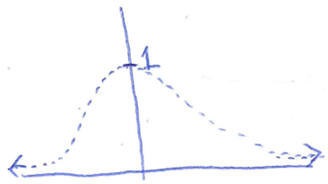
   b.) $\begin{cases} f\colon \mathbb{Z} \to \mathbb{Z} \\ f(n) = n + 1 \end{cases}$

   Solution: The function is one-to-one.



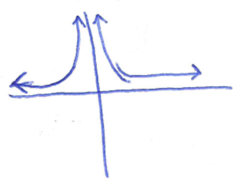
c.) $\begin{cases} f\colon \mathbb{Q} \to \mathbb{Q} \\ f(n) = \dfrac{1}{x^2+1} \end{cases}$

Solution: The function is **not** one-to-one.



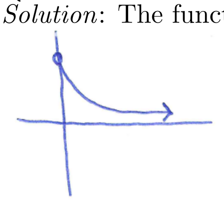
d.) $\begin{cases} f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R} \\ f(x) = \dfrac{1}{x^2} \end{cases}$

Solution: The function is **not** one-to-one.



e.) $\begin{cases} f\colon [0,\infty) \to \mathbb{R} \\ f(x) = \dfrac{1}{1+|x|} \end{cases}$

Solution: The function is one-to-one.



2. Show that $|\mathbb{Z}| = |\mathbb{N}|$.
   Solution: Define a function $f\colon \mathbb{Z} \to \mathbb{N}$ by

$$f(n) = \begin{cases} 0, & n = 0 \\ 2m-1, & n \text{ is the } m\text{th positive integer} \\ 2m, & n \text{ is the } m\text{th negative integer} \end{cases}$$

   Then $f(0) = 0$, $f(1) = 1$, $f(-1) = 2$, $f(2) = 3$, $f(-2) = 4$, $f(3) = 5$, $f(-3) = 6\ldots$. It maps 0 to 0, it maps positive integers to odd numbers, and it maps negative integers to even numbers. This shows $|\mathbb{Z}| \leq |\mathbb{N}|$. Defining a function $g\colon \mathbb{N} \to \mathbb{Z}$ by $g(n) = n$ is one-to-one, showing that $|\mathbb{N}| \leq |\mathbb{Z}|$.

   Therefore $|\mathbb{Z}| \leq |\mathbb{N}|$ and $|\mathbb{N}| \leq |\mathbb{Z}|$. Thus $|\mathbb{Z}| = |\mathbb{N}|$.

3. Show that $|\mathbb{N} \times \mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$. (Recall how the "$\times$" symbol works:

$$A \times B \times C = \{(a, b, c) \colon a \in A, b \in B, c \in C\}.)$$

   Solution: Define a function $f \colon \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by $f(a, b, c) = 2^a 3^b 5^c$. This function is one-to-one. Define a function $g \colon \mathbb{N} \to \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ by $g(n) = (n, 0, 0)$. This function is one-to-one. Thus $|\mathbb{N} \times \mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$.

4. Show that $|(0, 1]| = |[1, \infty)|$ (note: these are intervals on the real line). (*hint: consider rational functions*)
   Solution: Define a function $f \colon (0, 1] \to [1, \infty)$ by $f(x) = \dfrac{1}{x}$. This function is one-to-one. Define a function $g \colon [1, \infty) \to (0, 1]$ by $g(x) = \dfrac{1}{x}$. (*note*: the functions f and g are **different functions** because they have different domains and codomains!)

5. Show that $\left|\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)\right| = |\mathbb{R}|$. (*hint: consider inverse trigonometric functions*)
   Solution: The function $\tan \colon \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right) \to \mathbb{R}$ is one-to-one. The function $\arctan \colon \mathbb{R} \to \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ is one-to-one.

6. Find the cardinality of the set of functions $f \colon \{a, b\} \to \{c, d\}$ (i.e. how many such functions are there? brute force of listing all of them can get you the answer!).
   Solution: List all possible functions $f \colon \{a, b\} \to \{c, d\}$:
   1. $f(a) = c, f(b) = d$
   2. $f(a) = c, f(b) = c$
   3. $f(a) = d, f(b) = d,$
   4. $f(a) = d, f(b) = c.$

   Therefore the cardinality of the set of these functions is 4.