Homework 9 — MATH 2510 Spring 2018

Recall that the notation $f\colon A \to B$ is how you say that there is a function named "$f$" whose domain (i.e. set of inputs) is "$A$" and whose codomain (i.e. space where outputs live) is "$B$". Also recall that the range of a function is the set of actual outputs, i.e.

$$\text{range}(f) = \{y \in B \colon \exists x f(x) = y\}.$$

We defined a function $f\colon X \to Y$ to be one-to-one provided that $\forall x \forall y (f(x) = f(y) \to x = y)$. We defined the cardinality (i.e. "number of elements") $|A|$ of a set $A$ in the following way: we say that $|X| \leq |Y|$ provided there is a one-to-one function $f\colon X \to Y$. We say that $|X| = |Y|$ (i.e. "the cardinality of $X$ is equal to the cardinality of $Y$") provided that $|X| \leq |Y|$ and $|Y| \leq |X|$. Of course if a set is finite, we simply report its size as an number; e.g. $|\{a, b, c\}| = 3$.

1. Sketch each function (to the best of your abilities). Is the function one-to-one or not?

   a.) $\begin{cases} f\colon \mathbb{N} \to \mathbb{N} \\ f(n) = n + 1 \end{cases}$

   b.) $\begin{cases} f\colon \mathbb{Z} \to \mathbb{Z} \\ f(n) = n + 1 \end{cases}$

   c.) $\begin{cases} f\colon \mathbb{Q} \to \mathbb{Q} \\ f(n) = \dfrac{1}{x^2 + 1} \end{cases}$

   d.) $\begin{cases} f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R} \\ f(x) = \dfrac{1}{x^2} \end{cases}$

   e.) $\begin{cases} f\colon [0, \infty) \to \mathbb{R} \\ f(x) = \dfrac{1}{1 + |x|} \end{cases}$

2. Show that $|\mathbb{Z}| = |\mathbb{N}|$.

3. Show that $|\mathbb{N} \times \mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$. (Recall how the "$\times$" symbol works:

   $$A \times B \times C = \{(a, b, c) \colon a \in A, b \in B, c \in C\}.)$$

4. Show that $|(0, 1]| = |[1, \infty)|$ (note: these are intervals on the real line). (*hint: consider rational functions*)

5. Show that $\left|\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right| = |\mathbb{R}|$. (*hint: consider inverse trigonometric functions*)

6. Find the cardinality of the set of functions $f\colon \{a, b\} \to \{c, d\}$ (i.e. how many such functions are there? brute force of listing all of them can get you the answer!).