Homework 3 (solution) – MATH 2510 Spring 2018

#1.6: Show that the following statements are equivalent:

1. $F \equiv G$,
2. $\models F \leftrightarrow G$, and
3. $(F \wedge \neg G) \vee (\neg F \wedge G)$ is unsatisfiable.

Solution: First we show that if $F \equiv G$, then $\models F \leftrightarrow G$:

F	G	$F \leftrightarrow G$
1	1	1
$\not 1$	$\not 0$	$\not 0$
$\not 0$	$\not 1$	$\not 0$
0	0	1

In other words, we see in those rows for which $F \equiv G$ (i.e. $F \models G$ and $G \models F$, which means "whenever F is 1 it follows that G is 1" and vice versa – so we ignore the middle two rows!!), it is the case that $\models F \leftrightarrow G$ (i.e. $F \leftrightarrow G$ is a tautology).
Now we show that if $\models F \rightarrow G$, then $(F \wedge \neg G) \vee (\neg F \wedge G)$:

F	G	$F \leftrightarrow G$	$\neg F$	$\neg G$	$F \wedge \neg G$	$\neg F \wedge G$	$(F \wedge \neg G) \vee (\neg F \wedge G)$
1	1	1	0	0	0	0	0
$\not 1$	$\not 0$	$\not 0$	$\not 0$	$\not 1$	$\not 1$	$\not 0$	$\not 1$
$\not 0$	$\not 1$	$\not 0$	$\not 1$	$\not 0$	$\not 0$	$\not 1$	$\not 1$
0	0	1	1	1	0	0	0

which shows that when considering the rows which make $F \leftrightarrow G$ a tautology, that $(F \wedge \neg G) \vee (\neg F \wedge G)$ is unsatisfiable.
Finally, we will show that if $(F \wedge \neg G) \vee (\neg F \wedge G)$ is unsatisfiable, then $F \equiv G$.

F	G	$(F \wedge \neg G) \vee (\neg F \wedge G)$
1	1	0
$\not 1$	$\not 0$	$\not 1$
$\not 0$	$\not 1$	$\not 1$
0	0	0

From this we can observe that $F \models G$, because in any row for which F is true, it is also true that G is true. Similarly, we observe that $G \models F$ because in any row for which G is true, it follows that F is true. Therefore $F \equiv G$.

 # 1.9: The Cut rule states that from the formulas $F \to G$ and $G \to H$, we can derive $F \to H$. Verify this rule by giving a formal proof.

Solution: We have the set of formulas $\mathcal{F} = \{F \to G, G \to H\}$. We want to show that $\mathcal{F} \vdash F \to H$.

Statement	Justification
1. $\mathcal{F} \vdash F \to G$	Assumption
2. $\mathcal{F} \vdash G \to H$	Assumption
3. $\mathcal{F} \cup \{F\} \vdash F$	Assumption
4. $\mathcal{F} \cup \{F\} \vdash G$	$\to$-elimination on lines 1 and 3
5. $\mathcal{F} \cup \{F\} \vdash H$	$\to$-elimination on lines on lines 2 and 4
6. $\mathcal{F} \vdash F \to H$	$\to$-introduction on line 5

 # 1.10:

(a) Let $\leftrightarrow$-Symmetry be the following rule:
Premise: $\mathcal{F} \vdash (F \leftrightarrow G)$
Conclusion: $\mathcal{F} \vdash (G \leftrightarrow F)$
Verify this rule by giving a formal proof.

(b) Give a formal proof demonstrating that $\{(F \leftrightarrow G)\} \vdash (\neg F \leftrightarrow \neg G)$.

Solution: First we write a proof for (a). We let $\mathcal{F} \vdash \{(F \leftrightarrow G)\}$ and we will prove that $\mathcal{F} \vdash \{(G \leftrightarrow F)\}$.

Statement	Justification
1. $\mathcal{F} \vdash F \leftrightarrow G$	Assumption
2. $\mathcal{F} \vdash F \rightarrow G$	$\leftrightarrow$-definition
3. $\mathcal{F} \vdash G \rightarrow F$	$\leftrightarrow$-definition
4. $\mathcal{F} \vdash G \leftrightarrow F$	$\leftrightarrow$-definition on lines 3 and 2

Now we will write a proof for (b). Let $\mathcal{F} = \{(F \leftrightarrow G)\}$. We want to prove that $\mathcal{F} \vdash (\neg F \leftrightarrow \neg G)$. (*next page*)

Statement	Justification
1. $\mathcal{F} \vdash (F \leftrightarrow G)$	Assumption
2. $\mathcal{F} \vdash F \leftrightarrow G$	$(,)$-elimination on line 1
3. $\mathcal{F} \vdash F \rightarrow G$	$\leftrightarrow$-definition on line 2
4. $\mathcal{F} \cup \{F\} \vdash F$	Assumption
5. $\mathcal{F} \cup \{F\} \vdash F \rightarrow G$	Monotonicity on line 3
6. $\mathcal{F} \cup \{F\} \vdash G$	$\rightarrow$-elimination on lines 4 and 5
7. $\mathcal{F} \cup \{F\} \vdash \neg\neg G$	Double negation on line 6
8. $\mathcal{F} \vdash F \rightarrow \neg\neg G$	$\rightarrow$-introduction on line 7
9. $\mathcal{F} \vdash \neg F \vee \neg\neg G$	$\rightarrow$-definition on line 8
10. $\mathcal{F} \vdash \neg\neg G \vee \neg F$	$\vee$-symmetry on line 9
11. $\mathcal{F} \vdash \neg G \rightarrow \neg F$	$\rightarrow$-definition on line 10
12. $\mathcal{F} \vdash G \rightarrow F$	$\leftrightarrow$-definition on line 2
13. $\mathcal{F} \cup \{G\} \vdash G$	Assumption
14. $\mathcal{F} \cup \{G\} \vdash F$	$\rightarrow$-elimination on lines 12 and 13
15. $\mathcal{F} \cup \{G\} \vdash \neg\neg F$	Double negation on line 14
16. $\mathcal{F} \vdash G \rightarrow \neg\neg F$	$\rightarrow$-introduction on line 15
17. $\mathcal{F} \vdash \neg G \vee \neg\neg F$	$\rightarrow$-definition on line 16
18. $\mathcal{F} \vdash \neg\neg F \vee \neg G$	$\vee$-symmetry on line 17
19. $\mathcal{F} \vdash \neg F \rightarrow \neg G$	$\rightarrow$-definition on line 18
20. $\mathcal{F} \vdash \neg F \leftrightarrow \neg G$	$\leftrightarrow$-definition on lines 11 and 19
21. $\mathcal{F} \vdash (\neg F \leftrightarrow \neg G)$	$(,)$-introduction on line 20

 # 1.12: Prove the rule of "$\to$-contrapositive" with a formal proof:
Premise: $\mathcal{F} \vdash F \to G$
Conclusion: $\mathcal{F} \vdash \neg G \to \neg F$
Solution: Let $\mathcal{F} = \{F \to G\}$. Now prove

Statement	Justification
1. $\mathcal{F} \vdash F \to G$	Assumption
2. $\mathcal{F} \cup \{F\} \vdash F \to G$	Monotonicity on line 1
3. $\mathcal{F} \cup \{F\} \vdash F$	Assumption
4. $\mathcal{F} \cup \{F\} \vdash G$	$\to$-elimination on lines 2 and 3
5. $\mathcal{F} \cup \{F\} \vdash \neg\neg G$	Double negative of line 4
6. $\mathcal{F} \vdash F \to \neg\neg G$	$\to$-introduction on line 5
7. $\mathcal{F} \vdash \neg F \vee \neg\neg G$	$\to$-definition
8. $\mathcal{F} \vdash \neg\neg G \vee \neg F$	$\vee$-symmetry
9. $\mathcal{F} \vdash \neg G \to \neg F$	$\to$-definition

Problem A: Write a formal proof that shows $\{P, P \to Q\} \vdash Q$.
Solution: Let $\mathcal{F} = \{P, P \to Q\}$.

Statement	Justification
1. $\mathcal{F} \vdash P$	Assumption
2. $\mathcal{F} \vdash P \to Q$	Assumption
3. $\mathcal{F} \vdash Q$	$\to$-elimination using lines 1 and 2

Problem B: Write a formal proof showing $\{P \to Q, Q \to R, S \vee \neg R, P\} \vdash S$.
Solution: Let $\mathcal{F} = \{P \to Q, Q \to R, S \vee \neg R, P\}$.

Statement	Justification
1. $\mathcal{F} \vdash P \to Q$	Assumption
2. $\mathcal{F} \vdash Q \to R$	Assumption
3. $\mathcal{F} \vdash S \vee \neg R$	Assumption
4. $\mathcal{F} \vdash P$	Assumption
5. $\mathcal{F} \vdash \neg R \vee S$	$\vee$-symmetry on line 3
6. $\mathcal{F} \vdash R \to S$	$\to$-definition on line 5
7. $\mathcal{F} \vdash Q$	$\to$-elimination on lines 1 and 4
8. $\mathcal{F} \vdash R$	$\to$-elimination on lines 2 and 7
9. $\mathcal{F} \vdash S$	$\to$-elimination on lines 6 and 8