Homework 2 (solution) – MATH 2510 Spring 2018

1.1: Show that $\neg$ and $\vee$ can be taken as primitive symbols in propositional logic. That is, show that each of the symbols $\wedge, \rightarrow$, and $\leftrightarrow$ can be defined in terms of $\neg$ and $\vee$.

Solution: We can define $A \wedge B$ as $\neg(\neg A \vee \neg B)$. To see this, consider the following truth table:

A	B	$\neg A$	$\neg B$	$\neg A \vee \neg B$	$\neg(\neg A \vee \neg B)$	$A \wedge B$
1	1	0	0	0	1	1
1	0	0	1	1	0	0
0	1	1	0	1	0	0
0	0	1	1	1	0	0

We can define $A \rightarrow B$ as $B \vee \neg A$. To see this, consider the following truth table:

A	B	$\neg A$	$B \vee \neg A$	$A \rightarrow B$
1	1	0	1	1
1	0	0	0	0
0	1	1	1	1
0	0	1	1	1

The symbol $\leftrightarrow$ is normally defined as $(A \rightarrow B) \wedge (B \rightarrow A)$, meaning we can define it as $\neg(\neg(A \rightarrow B) \vee \neg(B \rightarrow A))$, in other words, as $\neg(\neg(B \vee \neg A) \vee \neg(A \vee \neg B))$. To see this, consider the following truth table:

A	B	$\neg A$	$\neg B$	$B \vee \neg A$	$A \vee \neg B$	$\neg(B \vee \neg A)$	$\neg(A \vee \neg B)$	$\neg(B \vee \neg A) \vee \neg(A \vee \neg B)$
1	1	0	0	1	1	0	0	0
1	0	0	1	0	1	1	0	1
0	1	1	0	1	0	0	1	1
0	0	1	1	1	1	0	0	0

(continued below)

$\neg(\neg(B \vee \neg A) \vee \neg(A \vee \neg B))$	$A \leftrightarrow B$
1	1
0	0
0	0
1	1

1.3: Find the truth tables for each of the following formulas. State whether each is a tautology, a contradiction, or neither.

(a) $(\neg A \rightarrow B) \vee ((A \wedge \neg C) \leftrightarrow B)$
Solution: Compute

A	B	C	$\neg A$	$\neg C$	$\neg A \rightarrow B$	$A \wedge \neg C$	$(A \wedge \neg C) \leftrightarrow B$	$(\neg A \rightarrow B) \vee ((A \wedge \neg C) \leftrightarrow B)$
1	1	1	0	0	1	0	0	1
1	1	0	0	1	1	1	1	1
1	0	1	0	0	1	0	1	1
1	0	0	0	1	1	1	0	1
0	1	1	1	0	1	0	0	1
0	1	0	1	1	1	0	0	1
0	0	1	1	0	0	0	1	1
0	0	0	1	1	0	0	1	1

Therefore we see that $(\neg A \rightarrow B) \vee ((A \wedge \neg C) \leftrightarrow B)$ is a tautology.

(b) $(A \to B) \wedge (A \to \neg B)$
Solution: Compute

A	B	$\neg B$	$A \to B$	$A \to \neg B$	$(A \to B) \wedge (A \to \neg B)$
1	1	0	1	0	0
1	0	1	0	1	0
0	1	0	1	1	1
0	0	1	1	1	1

Therefore we see that $(A \to B) \wedge (A \to \neg B)$ is neither a tautology nor a contradiction.

(c) $(A \to (B \vee C)) \vee (C \to \neg A)$
Solution: Calculate

A	B	C	$B \vee C$	$A \to (B \vee C)$	$\neg A$	$C \to \neg A$	$(A \to (B \vee C)) \vee (C \to \neg A)$
1	1	1	1	1	0	0	1
1	1	0	1	1	0	1	1
1	0	1	1	1	0	0	1
1	0	0	0	0	0	1	1
0	1	1	1	1	1	1	1
0	1	0	1	1	1	1	1
0	0	1	1	1	1	1	1
0	0	0	0	1	1	1	1

(d) $((A \to B) \land C) \lor (A \land D)$

Solution: Compute

A	B	C	D	$A \to B$	$(A \to B) \land C$	$A \land D$	$((A \to B) \land C) \lor (A \land D)$
1	1	1	1	1	1	1	1
1	1	1	0	1	1	0	1
1	1	0	1	1	0	1	1
1	1	0	0	1	0	0	0
1	0	1	1	0	0	1	1
1	0	1	0	0	0	0	0
1	0	0	1	0	0	1	1
1	0	0	0	0	0	0	0
0	1	1	1	1	1	0	1
0	1	1	0	1	1	0	1
0	1	0	1	1	0	0	0
0	1	0	0	1	0	0	0
0	0	1	1	1	1	0	1
0	0	1	0	1	1	0	1
0	0	0	1	1	0	0	0
0	0	0	0	1	0	0	0

From this we see that it is neither a tautology nor a contradiction.

1.4: In each of the following, determine whether the two formulas are equivalent.

(a) $(A \wedge B) \vee C$ and $(A \to \neg B) \to C$
Solution: It is sufficient to check if $((A \wedge B) \vee C) \leftrightarrow ((A \to \neg B) \to C)$ is a tautology. So, compute

A	B	C	$A \wedge B$	$(A \wedge B) \vee C$	$\neg B$	$A \to \neg B$	$(A \to \neg B) \to C$
1	1	1	1	1	0	0	1
1	1	0	1	1	0	0	1
1	0	1	0	1	1	1	1
1	0	0	0	0	1	1	0
0	1	1	0	1	0	1	1
0	1	0	0	0	0	1	0
0	0	1	1	1	1	1	1
0	0	0	0	0	1	1	0

Since the columns for $(A \wedge B) \vee C$ and $(A \to \neg B) \to C$ have the same values, we see that $((A \wedge B) \vee C) \leftrightarrow ((A \to \neg B) \to C)$ is a tautology (*note: I could have made this a column in the truth table itself, but I didn't have room to fit it in nicely!*)

(b) $(((A \to B) \to B) \to B)$ and $(A \to B)$
Solution: It is sufficient to check if $((((A \to B) \to B) \to B)) \leftrightarrow (A \to B)$ is a tautology. Compute

A	B	$A \to B$	$(A \to B) \to B$	$((A \to B) \to B) \to B$
1	1	1	1	1
1	0	0	1	0
0	1	1	1	1
0	0	1	0	1

Since the columns for $A \to B$ and $((A \to B) \to B) \to B$ are the same, we see that $((((A \to B) \to B) \to B)) \leftrightarrow (A \to B)$ is a tautology.

(c) $(((A \to B) \to A) \to A)$ and $(C \to D) \vee C$
Solution: It is sufficient to check if
$(((A \to B) \to A) \to A) \leftrightarrow ((C \to D) \vee C)$ is a tautology. Compute

A	B	C	D	$A \to B$	$(A \to B) \to A$	$((A \to B) \to A) \to A$	$C \to D$	$(C \to D) \vee C$
1	1	1	1	1	1	1	1	1
1	1	1	0	1	1	1	0	1
1	1	0	1	1	1	1	1	1
1	1	0	0	1	1	1	1	1
1	0	1	1	0	1	1	1	1
1	0	1	0	0	1	1	0	1
1	0	0	1	0	1	1	1	1
1	0	0	0	0	1	1	1	1
0	1	1	1	1	0	1	1	1
0	1	1	0	1	0	1	0	1
0	1	0	1	1	0	1	1	1
0	1	0	0	1	0	1	1	1
0	0	1	1	1	0	1	1	1
0	0	1	0	1	0	1	0	1
0	0	0	1	1	0	1	1	1
0	0	0	0	1	0	1	1	1

Since the column for $((A \to B) \to A) \to A$ and the column for
$(C \to C) \vee C$ are the same, the formula
$(((A \to B) \to A) \to A) \leftrightarrow ((C \to C) \vee C)$ is a tautology.

(d) $A \leftrightarrow ((\neg A \wedge B) \vee (A \wedge \neg B))$ and $\neg B$
Solution: It is sufficent to check if $(A \leftrightarrow ((\neg A \wedge B) \vee (A \wedge \neg B))) \leftrightarrow \neg B$ is a tautology. Compute

A	B	$\neg A$	$\neg B$	$\neg A \wedge B$	$A \wedge \neg B$	$(\neg A \wedge B) \vee (A \wedge \neg B)$	$A \leftrightarrow ((\neg A \wedge B) \vee (A \wedge \neg B))$
1	1	0	0	0	0	0	0
1	0	0	1	0	1	1	1
0	1	1	0	1	0	1	0
0	0	1	1	0	0	0	1

Since the columns for $A \leftrightarrow ((\neg A \wedge B) \vee (A \wedge \neg B))$ and for $\neg B$ are the same, $(A \leftrightarrow ((\neg A \wedge B) \vee (A \wedge \neg B))) \leftrightarrow \neg B$ is a tautology.

1.5: Show that the following statements are equivalent.

(a) $F \vDash G$,

(b) $\vDash F \rightarrow G$,

(c) $F \wedge \neg G$ is unsatifiable, and

(d) $F \equiv F \wedge G$.

Solution: First we show "if (a) holds, then (b) holds": if $F \vDash G$ is true, it means in any row where there is a "1" below F, then there is a "1" below G. We are trying to show that, in that circumstance, that $\vDash F \rightarrow G$, i.e. that $F \rightarrow G$ is a tautology. (*note: below, the second row is struck out because it does not satisfy that $F \vDash G$ since a 1 in the first column has a 0 in the second column...it is not under consideration!*)

F	G	$F \rightarrow G$
1	1	1
~~1~~	~~0~~	~~0~~
0	1	1
0	0	1

Notice that in all the rows under consideration that $F \rightarrow G$ is true, hence it is a tautology (in the circumstance in question, not in general!).
Now we show that "if (b) holds, then (c) holds": if $\vDash F \rightarrow G$, it means that the only rows under $F \rightarrow G$ under consideration are those with a 1 in them and $F \wedge \neg G$ being unsatisfiable means that the relevant rows in that column are all false. Compute

F	G	$F \to G$	$\neg G$	$F \wedge \neg G$
1	1	1	0	0
~~1~~	~~0~~	~~0~~	~~1~~	~~1~~
0	1	1	0	0
0	0	1	1	0

Notice that in all the rows under consideration that $F \wedge \neg G$ is false, hence it is unsatisfiable (in the circumstance in question!).
Now we show that "if (c) holds, then (d) holds": we must show that in the situation where $F \wedge \neg G$ is unsatisfiable, it follows that $F \equiv F \wedge G$, i.e. we must show that $F \leftrightarrow (F \wedge G)$ is a tautology. Compute

F	G	$\neg G$	$F \wedge \neg G$	$F \wedge G$	$F \leftrightarrow (F \wedge G)$
1	1	0	0	1	1
~~1~~	~~0~~	~~1~~	~~1~~	~~0~~	~~0~~
0	1	0	0	0	1
0	0	1	0	0	1

Notice that in all of the rows under consideration that $F \leftrightarrow (F \wedge G)$ is true, hence a tautology (in the circumstance in question!).
Finally, we show that "if (d) holds, then (a) holds". It is sufficient to show that in the sitaution where $F \leftrightarrow (F \wedge G)$ is a tautology, it follows that $F \vDash G$, i.e. that $F \to G$ is a tautology. Compute

F	G	$F \wedge G$	$F \leftrightarrow (F \wedge G)$	$F \to G$
1	1	1	1	1
~~1~~	~~0~~	~~0~~	~~0~~	~~0~~
0	1	0	1	1
0	0	0	1	1

Notice that in all rows under consideration that $F \to G$ is true, hence a tautology (in the circumstance in question!).
This completes the problem.

Prolog exercise. Consider the Prolog code `logicops.pl` at https://github.com/tomcuchta/math2510spring2018/blob/master/logicops.pl. Copy this code to SWISH.

The fact `p(a)` is given in the code. To write the conjunction "$A \wedge B$" in Prolog, we write `A,B` – this is why `conj` is defined as it is in the code. The negation symbol $\neg$ is written as `\+`. Definition 1.6 in the text defines the disjunction $\vee$ as in $P \vee Q$ as $\neg(\neg P \wedge \neg Q)$. This justifies the defintion of `disj` in the code.

1. Run the query `p(a)`. What is the result?
   Solution: `True`
2. Run the query `\+ p(a)`. What is the result?
   Solution: `False`
3. Run the query `p(b)`. What is the result?
   Solution: `False`
4. Run the query `\+ p(b)`. What is the result?
   Solution: `True`
5. Define `impl` to encode the symbol $\rightarrow$ as in $P \rightarrow Q$ in terms of `disj`.
   Solution: `impl(X,Y) :- disj(Y,\+ X).`
6. Define `iff` to encode the symbol $\leftrightarrow$ as in $P \longleftrightarrow Q$ in terms of `impl`.
   Solution: `iff(X,Y) :- conj(impl(X,Y),impl(Y,X)).`
7. Run `impl(p(a),p(b))`. What is the result?
   Solution: `False`
8. Run `impl(\+ p(a),p(b))`. What is the result?
   Solution: `True`
9. Run `iff(p(a),p(b))`. What is the result?
   Solution: `False`
10. Run `iff(p(b),p(a))`. What is the result?
    Solution: `False`