MATH 2510 - EXAM 1 - SPRING 2018
SOLUTION

8 February 2018
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Instructions:

- Show all work, clearly and in order, if you want to get full credit. If you claim something is true **you must show work backing up your claim**. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers algebraically whenever possible to ensure full credit.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point.
- Good luck!

1. (15 points) Let $X = \{1, 2\}$, $Y = \{4, 6\}$, and $Z = \{1, 9\}$.
   (a) (3 points) Compute $X \times Y$.
   Solution: $X \times Y = \{(1, 4), (1, 6), (2, 4), (2, 6)\}$
   (b) (3 points) Compute $Z \times Y$.
   Solution: $Z \times Y = \{(1, 4), (1, 6), (9, 4), (9, 6)\}$
   (c) (3 points) Using your answers from above, what is $(X \times Y) \cup (Z \times Y)$?
   Solution: $(X \times Y) \cup (Z \times Y) = \{(1, 4), (1, 6), (2, 4), (2, 6), (9, 4), (9, 6)\}$
   (d) (3 points) Using your answers from above, what is $(X \times Y) \cap (Z \times Y)$?
   Solution: $(X \times Y) \cap (Z \times Y) = \{(1, 4), (1, 6)\}$
   (e) (3 points) Using your answer from above, is $(X \times Y)$ a function?
   Solution: No – it contains the points $(1, 4)$ and $(1, 6)$ and therefore fails the "vertical line test".

2. (16 points) Formula or not? If not, circle the problem(s). Find all errors for full credit.
   (a) (4 points) $P \leftrightarrow (P \vee Q \rightarrow \neg P)$
   Solution: Formula
   (b) (4 points) $(P \neg \wedge Q) \rightarrow \neg(P \neg Q)$
   Solution: Not a formula
   (c) (4 points) $\leftrightarrow P \rightarrow Q$
   Solution: Not a formula
   (d) (4 points) $P \leftrightarrow (P \vee Q \rightarrow \neg(\neg P))$
   Solution: Formula

3. (7 points) Find a truth table for the formula $P \leftrightarrow (P \vee \neg Q)$.
Solution: Calculate

P	Q	$\neg Q$	$P \vee \neg Q$	$P \leftrightarrow (P \vee \neg Q)$
1	1	0	1	1
1	0	1	1	1
0	1	0	0	1
0	0	1	1	0

4. (7 points) Is $(P \rightarrow Q) \equiv (\neg P \rightarrow \neg Q)$? Check it by making an appropriate truth table.
Solution: We know that to show a formula G is a consequence of a formula F, i.e. $F \equiv G$, it is sufficient to show that $F \leftrightarrow G$ is a tautology. If it is not a tautology, then they are not equivalent. Compute

P	Q	$P \rightarrow Q$	$\neg P$	$\neg Q$	$\neg P \rightarrow \neg Q$	$(P \rightarrow Q) \leftrightarrow (\neg P \rightarrow \neg Q)$
1	1	1	0	0	1	1
1	0	0	0	1	1	0
0	1	1	1	0	0	0
0	0	1	1	1	1	1

From this, we see that $(P \rightarrow Q) \not\equiv (\neg P \rightarrow \neg Q)$.

5. (12 points) Satisfiable or unsatisfiable?

   (a) (6 points) $P \to (Q \to \neg P)$
   Solution: To check if it is satisfiable or not, it is sufficient to construct a truth table for the formula and observe if there is a "1" in its column. Compute

P	Q	$\neg P$	$Q \to \neg P$	$P \to (Q \to \neg P)$
1	1	0	0	0
1	0	0	1	1
0	1	1	1	1
0	0	1	1	1

   From this, we see that $P \to (Q \to \neg P)$ is satisfiable.

   (b) (6 points) $(P \leftrightarrow Q) \leftrightarrow (\neg P \leftrightarrow Q)$
   Solution: To check if it is satisfiable or not, it is sufficient to construct a truth table for the formula and observe if there is a "1" in its column. Compute

P	Q	$\neg P$	$P \leftrightarrow Q$	$\neg P \leftrightarrow Q$	$(P \leftrightarrow Q) \leftrightarrow (\neg P \leftrightarrow Q)$
1	1	0	1	0	0
1	0	0	0	1	0
0	1	1	0	1	0
0	0	1	1	0	0

   From this, we see that $(P \leftrightarrow Q) \leftrightarrow (\neg P \leftrightarrow Q)$ is unsatisfiable.

6. (18 points) (a) (6 points) Consider the assignment $\mathcal{A}(P) = 1$ and $\mathcal{A}(Q) = 0$. Does $\mathcal{A} \vDash P \vee Q$?
   Solution: Yes – this is because the "or" of a true thing (that is, P) and a false thing (that is, Q) is true.

   (b) (6 points) Is the following correct to write?: $\vDash P \wedge \neg P$. Why or why not?
   Solution: No. This is because $P \wedge \neg P$ is a contradiction. To see that, compute

P	$\neg P$	$P \wedge \neg P$
1	0	0
0	1	0

   (c) (6 points) Use a truth table to show that $P \wedge Q \vDash P$.
   Solution: It is sufficient to show that $P \wedge Q \to P$ is a tautology, i.e. in a truth table, that formula consists of a column of "1"s. Compute

P	Q	$P \wedge Q$	$P \wedge Q \to P$
1	1	1	1
1	0	0	1
0	1	0	1
0	0	0	1

   From this we observe that $P \wedge Q \to P$ is a tautology, and hence $P \wedge Q \vDash P$.

7. (25 points) (a) (12 points) Let $\mathcal{F} = \{P, Q \vee \neg P\}$ and show that $\mathcal{F} \vdash Q$ by writing a formal proof.
   Solution: Write the proof!

Statement	Justification
1. $\mathcal{F} \vdash P$	Assumption
2. $\mathcal{F} \vdash Q \vee \neg P$	Assumption
3. $\mathcal{F} \vdash \neg P \vee Q$	$\vee$-symmetry on line 2
4. $\mathcal{F} \vdash P \to Q$	$\to$-definition on line 3
5. $\mathcal{F} \vdash Q$	$\to$-elimination on lines 1 and 4

(b) (13 points) Let $\mathcal{F} = \{P \leftrightarrow Q, \neg\neg P \rightarrow H, Q\}$ and show that $\mathcal{F} \vdash H$ by writing a formal proof.

Solution: Write the proof!

Statement	Justification
1. $\mathcal{F} \vdash P \leftrightarrow Q$	Assumption
2. $\mathcal{F} \vdash Q \rightarrow P$	$\leftrightarrow$-definition on line 1
3. $\mathcal{F} \vdash Q$	Assumption
4. $\mathcal{F} \vdash P$	$\rightarrow$-elimination on lines 2 and 3
5. $\mathcal{F} \vdash \neg\neg P$	Double negation on line 4
6. $\mathcal{F} \vdash \neg\neg P \rightarrow H$	Assumption
7. $\mathcal{F} \vdash H$	$\rightarrow$-elimination on lines 5 and 6

Table 1.5 Basic rules for derivations

Premise	Conclusion	Name
G is in $\mathcal{F}$	$\mathcal{F} \vdash G$	Assumption
$\mathcal{F} \vdash G$ and $\mathcal{F} \subset \mathcal{F}'$	$\mathcal{F}' \vdash G$	Monotonicity
$\mathcal{F} \vdash G$	$\mathcal{F} \vdash \neg\neg G$	Double negation
$\mathcal{F} \vdash F$, $\mathcal{F} \vdash G$	$\mathcal{F} \vdash (F \wedge G)$	$\wedge$-Introduction
$\mathcal{F} \vdash (F \wedge G)$	$\mathcal{F} \vdash F$	$\wedge$-Elimination
$\mathcal{F} \vdash (F \wedge G)$	$\mathcal{F} \vdash (G \wedge F)$	$\wedge$-Symmetry
$\mathcal{F} \vdash F$	$\mathcal{F} \vdash (F \vee G)$	$\vee$-Introduction
$\mathcal{F} \vdash (F \vee G)$, $\mathcal{F} \cup \{F\} \vdash H$, $\mathcal{F} \cup \{G\} \vdash H$	$\mathcal{F} \vdash H$	$\vee$-Elimination
$\mathcal{F} \vdash (F \vee G)$	$\mathcal{F} \vdash (G \vee F)$	$\vee$-Symmetry
$\mathcal{F} \cup \{F\} \vdash G$	$\mathcal{F} \vdash (F \rightarrow G)$	$\rightarrow$-Introduction
$\mathcal{F} \vdash (F \rightarrow G)$, $\mathcal{F} \vdash F$	$\mathcal{F} \vdash G$	$\rightarrow$-Elimination
$\mathcal{F} \vdash F$	$\mathcal{F} \vdash (F)$	(,)-Introduction
$\mathcal{F} \vdash (F)$	$\mathcal{F} \vdash F$	(,)-Elimination
$\mathcal{F} \vdash ((F \wedge G) \wedge H)$	$\mathcal{F} \vdash (F \wedge G \wedge H)$	$\wedge$-Parentheses rule
$\mathcal{F} \vdash ((F \vee G) \vee H)$	$\mathcal{F} \vdash (F \vee G \vee H)$	$\vee$-Parentheses rule

Table 1.6 More rules for derivations

Rules	Name
$\mathcal{F} \vdash (F \vee G)$ if and only if $\mathcal{F} \vdash \neg(\neg F \wedge \neg G)$	$\vee$-Definition
$\mathcal{F} \vdash (F \rightarrow G)$ if and only if $\mathcal{F} \vdash (\neg F \vee G)$	$\rightarrow$-Definition
$\mathcal{F} \vdash (F \leftrightarrow G)$ if and only if both $\mathcal{F} \vdash (F \rightarrow G)$ and $\mathcal{F} \vdash (G \rightarrow F)$	$\leftrightarrow$-Definition