HW10 MATH 2501 FALL 2018

Section 4.5

#226

$f(x) = x^4 - 6x^3$

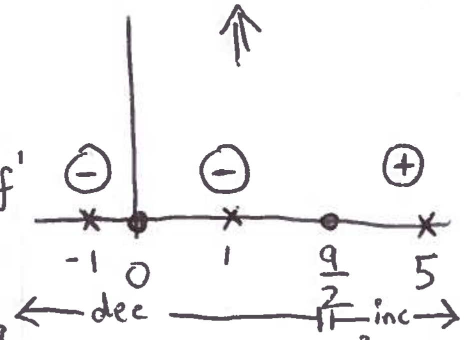
$f'(x) = \boxed{4x^3 - 18x^2 \overset{\text{set}}{=} 0}$

$x^2(4x - 18) = 0$

$x^2 = 0 \qquad 4x - 18 = 0$

$x = 0 \qquad x = \frac{18}{4} = \frac{9}{2}$

$\Rightarrow$



$\Uparrow$

$\boxed{\text{local min at } x = \frac{9}{2}}$

$f'(-1) = 4(-1)^3 - 18(-1)^2$

$= -4 - 18 < 0$

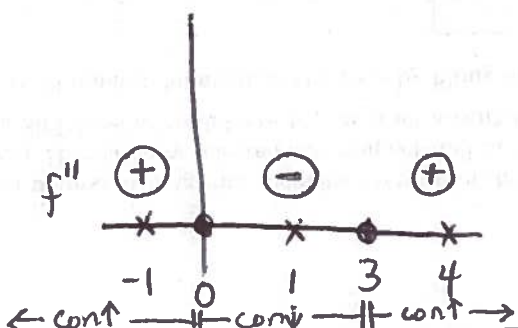
$f'(1) = 4 - 18 < 0$

$f'(5) = 4(5^3) - 18(5^2) > 0$

$f''(x) = \boxed{12x^2 - 36x \overset{\text{set}}{=} 0}$

$x(12x - 36) = 0$

$x = 0 \qquad x = \frac{36}{12} = 3$



$f''(-1) = 12(-1)^2 - 36(-1) > 0$

$f''(1) = 12(1)^2 - 36(1) < 0$

$f''(4) = 12(4^2) - 36(4) > 0$

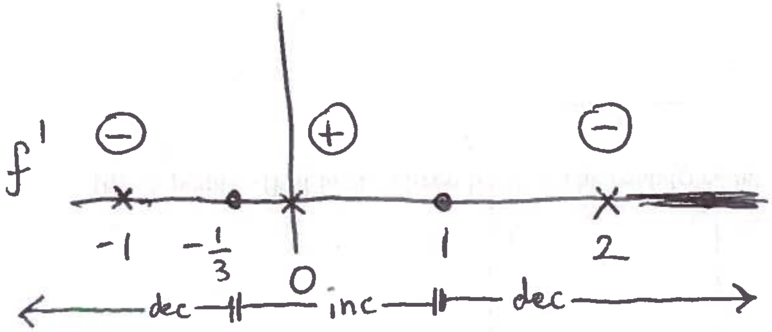
$\Rightarrow \boxed{\text{inflection points at } x = 0 \text{ and } x = 3}$

#228 $f(x) = x + x^2 - x^3$

$f'(x) = \boxed{1 + 2x - 3x^2 \overset{set}{=} 0}$

quadratic formula

$3x^2 - 2x - 1 = 0 \longrightarrow x = \dfrac{2 \pm \sqrt{4 - 4(3)(-1)}}{2(3)} = \dfrac{2 \pm \sqrt{16}}{6} = \dfrac{2 \pm 4}{6}$ $\overset{\oplus}{\nearrow} \dfrac{6}{6} = 1$ $\underset{\ominus}{\searrow} -\dfrac{2}{6} = -\dfrac{1}{3}$



$f'(-1) = 1 - 2 - 3 < 0$

$f'(0) = 1 > 0$

$f'(2) = 1 + 4 - 12 < 0$

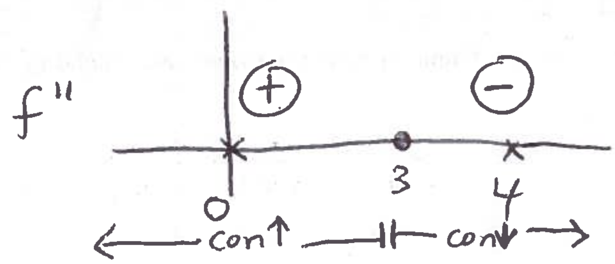
local min at $x = -1/3$

local max at $x = 1$

$f''(x) = \boxed{2 - 6x \overset{set}{=} 0} \longrightarrow x = 3$

$f''(0) = 2 > 0$

$f''(4) = 2 - 6(4) < 0$



inflection point at $x = 3$

#238

$f(x) = \ln(x)\sqrt{x} = \ln(x)\, x^{1/2}, \quad x > 0$

$f'(x) = \boxed{\frac{1}{x} x^{1/2} + \ln(x)\left(\frac{1}{2}\right) x^{-1/2} \overset{\text{set}}{=} 0} \longrightarrow f'(x) = \frac{1}{\sqrt{x}} + \frac{1}{2}\frac{\ln(x)}{\sqrt{x}}$

$\underbrace{\frac{\sqrt{x}}{x}}_{= \frac{1}{\sqrt{x}}} + \frac{1}{2}\frac{\ln(x)}{\sqrt{x}} = 0 \Rightarrow \frac{1}{\sqrt{x}} + \frac{1}{2}\frac{\ln(x)}{\sqrt{x}} = 0$

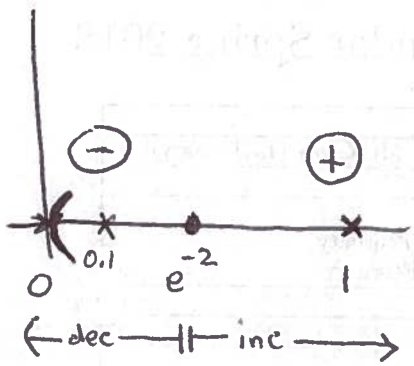
$\xRightarrow[\text{multiply by } \sqrt{x}]{\text{since } x>0,} 1 + \frac{1}{2}\ln(x) = 0$

$\frac{1}{2}\ln(x) = -1$

$\ln(x) = -2$

$\boxed{x = e^{-2}} \approx 0.13$

f'



local min at $x = e^{-2}$

$f'(0.1) = \frac{\sqrt{0.1}}{0.1} + \frac{\ln(0.1)\left(\frac{1}{2}\right)}{\sqrt{0.1}} \approx -0.47 < 0$

$f'(1) = 1 + \frac{\ln(1)\frac{1}{2}}{\sqrt{1}} = 1 + 0 > 0$

$f''(x) = \frac{d}{dx}\left[x^{-1/2} + \frac{1}{2}x^{-1/2}\ln(x)\right]$

$= \boxed{-\frac{1}{2}x^{-3/2} + \frac{1}{2}\left(-\frac{1}{2}x^{-3/2}\ln(x) + \frac{x^{-1/2}}{x}\right) \overset{\text{set}}{=} 0}$, where $\frac{x^{-1/2}}{x} = x^{-3/2}$ $\longrightarrow f''(x) = x^{-3/2}\left(-\frac{1}{2} - \frac{1}{4}\ln(x) + 1\right)$

$\downarrow$ multiply by $x^{3/2}$

$-\frac{1}{2} - \frac{1}{4}\ln(x) + 1 = 0$

$-\frac{1}{4}\ln(x) = -\frac{1}{2}$

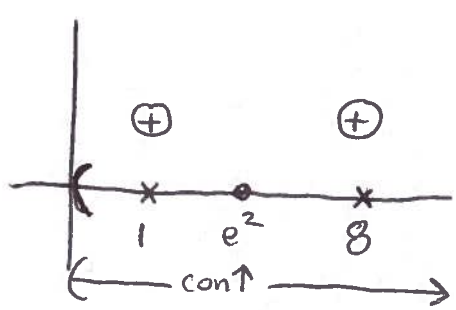
$\ln(x) = 2$

$x = e^2 \approx 7.38$

$f''(1) = 1\left(\frac{1}{2} - \frac{1}{4}(\overset{=0}{\widetilde{\ln(1)}})\right) = \frac{1}{2} > 0$

$f''(8) = 8^{-3/2}\left(1 - \frac{1}{4}\ln(8)\right) > 0$

f''



no inflection point

#240 $f(x)=\frac{e^x}{x},\ x\neq 0$

$$f'(x)=\frac{xe^x-e^x(1)}{x^2}=\boxed{\frac{e^x}{x}-\frac{e^x}{x^2}\overset{\text{set}}{=}0}$$

quotient Rule

$$e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)=0$$

divide

$e^x=0$ → NO SOLN (e^x never $=0$)

$\frac{1}{x}-\frac{1}{x^2}=0$

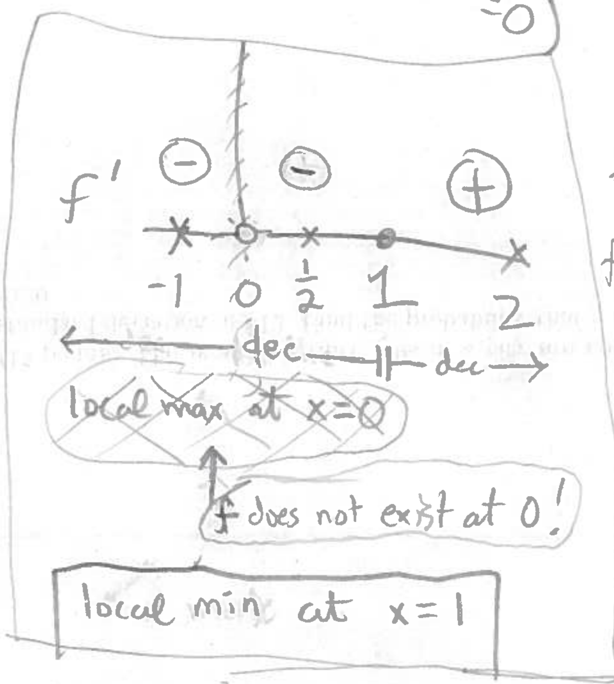
$\frac{1}{x}=\frac{1}{x^2}$

reciprocal

$x=x^2$

$0=x^2-x \rightarrow 0=x(x-1)$

$x=0$ $\quad x-1=0 \rightarrow x=1$



$f'(\frac{1}{2})=\frac{\sqrt{e}}{1/2}-\frac{\sqrt{e}}{1/4}=2\sqrt{e}-4\sqrt{e}<0$

$f'(-1)=\frac{e^{-1}}{(-1)}-\frac{e^{-1}}{(-1)^2}<0$

$f'(2)=\frac{e^2}{2}-\frac{e^2}{4}=\frac{e^2}{4}>0$

$$f''(x)=\boxed{\frac{xe^x-e^x(1)}{x^2}-\frac{x^2e^x-e^x(2x)}{x^4}\overset{\text{set}}{=}0}$$

$f''(-1)=\underbrace{\frac{-e^{-1}-e^{-1}}{(-1)^2}}_{=0}-\frac{(-1)^2e^{-1}+2e^{-1}}{(-1)^4}$ $\quad -3e^{-1}<0$

$f''(1)=\underbrace{\frac{e-e}{1}}_{=0}-\frac{e-2e}{1}=-e<0$

$$e^x\left(\frac{x-1}{x^2}-\frac{x^2-2x}{x^4}\right)=0$$

multiply by $\frac{x^4}{e^x}$

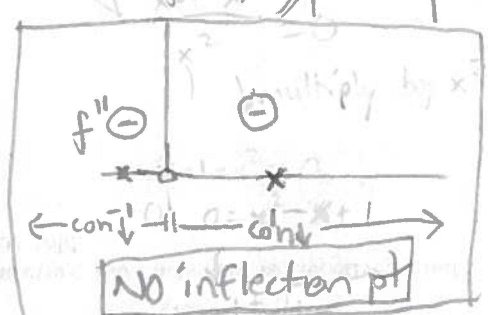
$x^2(x-1)-(x^2-2x)=0$

$x^3-x^2-x^2+2x=0$

$x(x^2-2x+2)=0$

divide by x $(x\neq 0)$

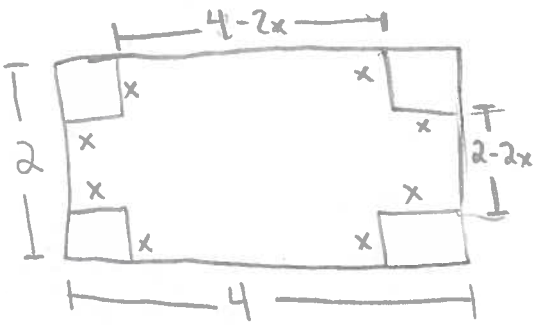
$x^2-2x+2=0$



Roots

$x=\frac{2\pm\sqrt{4-8}}{2}=1\pm 2i$ not real roots

#316



Physical restrictions

$0 < x < 1$ (because of side of length 2)

Volume of box: $V = (4-2x)(2-2x)x$

$$= (8-8x-4x+4x^2)x$$
$$= x(4x^2-12x+8)$$
$$= 4x^3-12x^2+8x$$
$$V' = \boxed{12x^2-24x+8 \underset{\text{set}}{=} 0}$$

$\downarrow$ divide by 4

$$3x^2-6x+2=0$$

36-24

$$x = \frac{6 \pm \sqrt{36-4(3)(2)}}{2(3)} = \frac{6 \pm \sqrt{12}}{6} = 1 \pm \frac{\sqrt{12}}{6}$$

The solution $1+\frac{\sqrt{12}}{6}$ not in our allowed region $0<x<1$.

Our only (physically meaningful) critical point is at $x = 1 - \frac{\sqrt{12}}{6} \approx 0.422$

To see that a max occurs there, calculate

$$V''(x) = 24x-24$$

$\downarrow$ plug in $x = 1-\frac{\sqrt{12}}{6}$

$$V''\left(1-\frac{\sqrt{12}}{6}\right) = 24\underbrace{\left(1-\frac{\sqrt{12}}{6}\right)}_{<1}-24 < 0$$

Therefore by second derivative test, we observe that there is a local max at $x = 1-\frac{\sqrt{12}}{6}$.

#318 two positive integers: call them x and y

"sum is 10" $\rightsquigarrow$ $x+y=10$ (i)

"sum of squares" $\rightsquigarrow$ x^2+y^2 (ii) ← minimize/ +maximize

Soln: From (i),

$$x = 10-y.$$

Plug that into (ii) to obtain

$$f(y) = (10-y)^2+y^2 =$$

$$f' = 4y-20=0$$

$$f'(y) = \boxed{\overbrace{2(10-y)(-1)+2y}^{f'=4y-20=0} \overset{\text{set}}{=} 0}$$

$$\Rightarrow 2y-20+2y=0$$

$$4y=20$$

$$y=5 \leftarrow \text{critical point}$$

$f''(y) = 4$,

so calculate $f''(5)=4>0$, thus by 2nd derivative test, f has a local min at $x=5$.

#320

$\longrightarrow$ $x+2y=800$ (i) ← total amount of fencing

Area $= A = xy$ (ii)

Solve (i) for x:

$$x = 800-2y$$

Plug into (ii):

$$A(y) = (800-2y)y = 800y-2y^2$$

$$A'(y) = \boxed{800-4y \overset{\text{set}}{=} 0}$$

$$800=4y$$

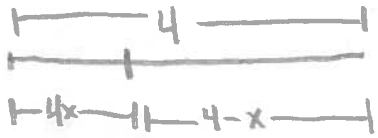
$$\boxed{200=y} \xrightarrow{\text{plug into (i)}} x+2(200)=800$$

$$\boxed{x=400}$$

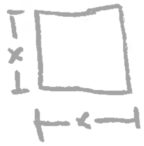
Prove it maximizes

$$A''=-4 \longrightarrow A''(200)<0 \longrightarrow \boxed{\text{local max at } y=-4}$$

#336

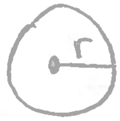


form square



Area $= x^2$

form circle



Area $= \pi r^2$

$= \pi\left(\frac{4-x}{2\pi}\right)^2 = \frac{(4-x)^2}{4\pi}$

note: Circle has circumference

$\underbrace{4-x}_{\text{wire}} = C = \underbrace{2\pi r}_{\text{geometry}}$

$\Downarrow$

$r = \frac{4-x}{2\pi}$

Sum of areas $= A(x) = x^2 + \frac{(4-x)^2}{4\pi}$

$\downarrow$

$A'(x) = 2x + \frac{2(4-x)(-1)}{\pi}$

$\boxed{= 2x + \frac{2x-8}{\pi} \overset{\text{set}}{=} 0}$

$x\left(2+\frac{2}{\pi}\right) = \frac{8}{\pi}$

$x = \frac{8/\pi}{2+2/\pi} = \frac{8}{2\pi+2}$

minimize/maximize?

$A'' = 2 + \frac{2}{\pi}$

$A''\left(\frac{8/\pi}{2+2/\pi}\right) > 0$

local min

BUT is it abmin?

Possible values for x: $0 \le x \le 4$

"all wire to circle" "all wire to square"

x	$A(x)$	
0	$0 + \frac{16}{4\pi} = 4/\pi \approx 1.273$	abmin
$\frac{8}{2\pi+2}$	$\left(\frac{8}{2\pi+2}\right)^2 + \frac{1}{4\pi}\left(4 - \frac{8}{2\pi+2}\right)^2 \approx 1.66$	
4	$4^2 + 0 \approx 16$	abmax

So area minimized when using all wire for square (i.e. $x=0$)
and maximized when using all wire for circle (i.e. $x=4$)