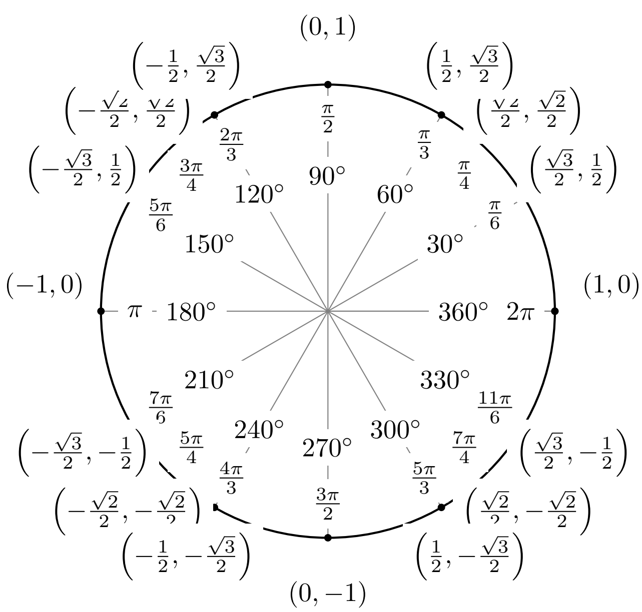
MATH 2501 - EXAM 4 - FALL 2018
SOLUTION

Friday 16 November 2018
Instructor: Tom Cuchta

Instructions:

- Show all work, clearly and in order, if you want to get full credit. If you claim something is true **you must show work backing up your claim**. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers algebraically whenever possible to ensure full credit.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point.
- Good luck!

Formulas



Newton's formula for solving the equation $f(x)=0$:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Definition of the definite integral of f on $[a,b]$:
if $\Delta x = \dfrac{b-a}{n}$, then

$$\int_a^b f(x)\mathrm{d}x = \lim_{n\to\infty} \sum_{k=1}^n f(a+k\Delta x)\Delta x$$

L'Hôpital:
if $L = \lim \dfrac{f(x)}{g(x)}$ yields indeterminant form, then $L = \lim \dfrac{f'(x)}{g'(x)}$

Special sums:

$$\sum_{k=1}^n k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^n k^3 = \frac{n^2(n+1)^2}{4}$$

1. (6 points) Calculate the limit in any way that you choose:

$$\lim_{x\to\infty} \frac{8x}{x-3}.$$

Solution: Since plugging in $x = \infty$ leads to indeterminant form $\frac{\infty}{\infty}$, we may apply L'hôpital's rule to compute

$$\lim_{x\to\infty} \frac{8x}{x-3} \overset{\text{L.H.}}{=} \lim_{x\to\infty} \frac{\frac{\mathrm{d}}{\mathrm{d}x}8x}{\frac{\mathrm{d}}{\mathrm{d}x}\left[x-3\right]} = \lim_{x\to\infty} \frac{8}{1} = 8.$$

2. (11 points) (a) (5 points) Write Newton's formula for solving the equation $e^x + \sin(x) = 0$.
Solution: Here $f(x) = e^x + \sin(x)$ and so $f'(x) = e^x + \cos(x)$. Therefore we get

$$x_{n+1} = x_n - \frac{e^{x_n} + \sin(x_n)}{e^{x_n} + \cos(x_n)}.$$

(b) (6 points) Write Newton's formula for solving the equation $\ln(x) = x^2 + 1$.
Solution: This given equation is not of the form $f(x) = 0$. Therefore subtract to get the equation $\ln(x) - x^2 - 1 = 0$. In this case we take $f(x) = \ln(x) - x^2 - 1$ and so $f'(x) = \frac{1}{x} - 2x$. Therefore we get

$$x_{n+1} = x_n - \frac{\ln(x_n) - x_n^2 - 1}{\frac{1}{x_n} - 2x_n}$$

3. (7 points) Find the values of x_1 and x_2 if $x_0 = 2$ and $x_{n+1} = 1 + x_n^2$.
Solution: Plugging in $n = 0$ into the recurrence yields

$$x_1 = 1 + x_0^2 \underbrace{=}_{x_0=2} 1 + 2^2 = 1 + 4 = 5.$$

Plugging $n = 1$ into the recurrence yields

$$x_2 = 1 + x_1^2 \underbrace{=} x_1 = 51 + 5^2 = 1 + 25 = 26.$$

4. (23 points) Answer the following questions about antiderivatives.
(a) (7 points) Find the antiderivative(s) of the function $f(x) = e^x + x^3$.
Solution: The antiderivatives are given by

$$D^{-1}\left[e^x + x^3\right] = D^{-1}\left[e^x\right] + D^{-1}\left[x^3\right] = e^x + \frac{x^4}{4} + C,$$

where C is an arbitrary constant.
(b) (8 points) Find the function f that obeys the properties $f'(x) = x^2$ and $f(0) = 3$.
Solution: Taking antiderivatives yields

$$f(x) = D^{-1}f'(x) = D^{-1}\left[x^2\right] = \frac{x^3}{3} + C.$$

Using the initial condition $f(0) = 3$, we observe

$$\underbrace{3}_{\text{given}} = f(0) = \underbrace{0 + C}_{\text{computed}}.$$

Therefore $C = 3$, and we have

$$f(x) = \frac{x^3}{3} + 3.$$

(c) (8 points) Find all functions f with the property that $f''(x) = x$.
Solution: Taking one antiderivative yields

$$f'(x) = D^{-1}\left[f''(x)\right] = D^{-1}\left[x\right] = \frac{x^2}{2} + C,$$

where C is an arbitrary constant. Taking a second antiderivative yields

$$f(x) = D^{-1}\left[f'(x)\right] = D^{-1}\left[\frac{x^2}{2} + C\right] = \frac{x^3}{6} + Cx + E,$$

where E is an arbitrary constant.

5. (6 points) Calculate $\sum_{k=7}^{9} k^2$.
Solution: Compute directly

$$\sum_{k=7}^{9} k^2 = \underbrace{7^2}_{k=7} + \underbrace{8^2}_{k=8} + \underbrace{9^2}_{k=9} = 49 + 64 + 81 = 194.$$

6. (8 points) Use the **definition of** the definite integral to compute $\int_0^2 x\mathrm{d}x$.
Solution: First note that $\Delta x = \dfrac{2-0}{n}$. Taking $f(x) = x$ we see that

$$\begin{aligned}
\int_0^2 x\mathrm{d}x &= \lim_{n\to\infty} \sum_{k=1}^{n} f\left(0 + k\Delta x\right)\Delta x \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{2k}{n}\right)\frac{2}{n} \\
&= \lim_{n\to\infty} \frac{4}{n^2} \underbrace{\sum_{k=1}^{n} k}_{=\frac{n(n+1)}{2}} \\
&= \frac{4}{2} \underbrace{\lim_{n\to\infty} \frac{n^2+n}{n^2}}_{=1} \\
&= 2.
\end{aligned}$$

7. (16 points) Calculate the integral in any way you wish.

(a) (8 points) $\int_0^5 x + 3\mathrm{d}x$
Solution: Using the fundamental theorem of calculus, compute

$$\int_0^5 x + 3\mathrm{d}x = \frac{x^2}{2} + 3x\Big|_0^5 = \left(\frac{5^2}{2} + 3(5)\right) - 0 = \frac{25}{2} + 15 = \frac{55}{2}.$$

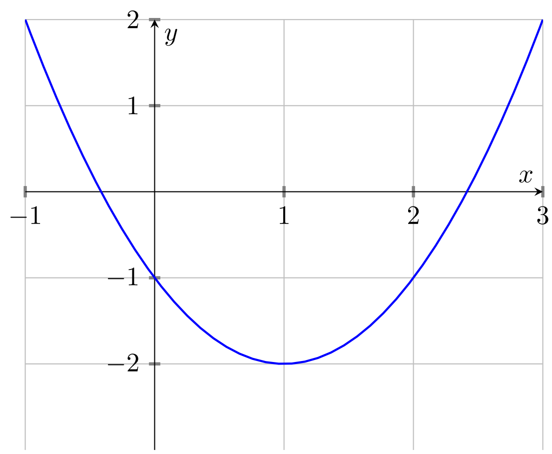
(b) (8 points) $\int_0^{\frac{\pi}{2}} \sin(x)\mathrm{d}x$
Solution: Using the fundamental theorem of calculus, compute

$$\int_0^{\frac{\pi}{2}} \sin(x)\mathrm{d}x = -\cos(x)\Big|_0^{\frac{\pi}{2}} = -\cos\left(\frac{\pi}{2}\right) - \left(-\cos(0)\right) = -0 + 1 = 1.$$

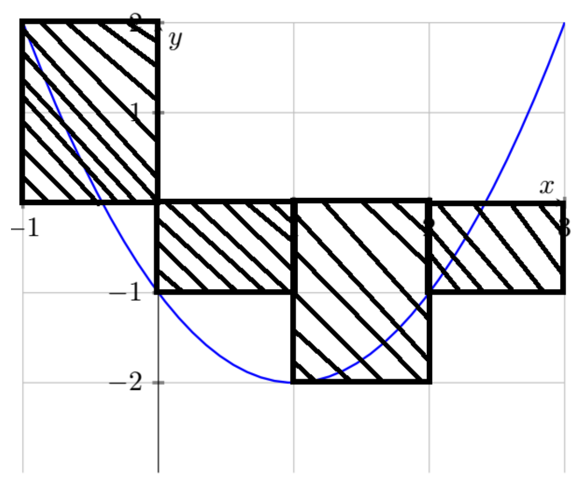
8. (7 points) If $\int_1^2 f(x)\mathrm{d}x = 3$ and $\int_2^5 f(x)\mathrm{d}x = -2$, then calculate $\int_1^5 f(x)\mathrm{d}x$.
Solution: Using the properties of integrals, compute

$$\int_1^5 f(x)\mathrm{d}x = \underbrace{\int_1^2 f(x)\mathrm{d}x}_{=3} + \underbrace{\int_2^5 f(x)\mathrm{d}x}_{=-2} = 3 + (-2) = 1.$$

9. (8 points) Draw the rectangles for the left endpoint approximation L_4 and use it to estimate the integral $\int_{-1}^3 f(x)\mathrm{d}x$ for the following graphed function $f(x)$.



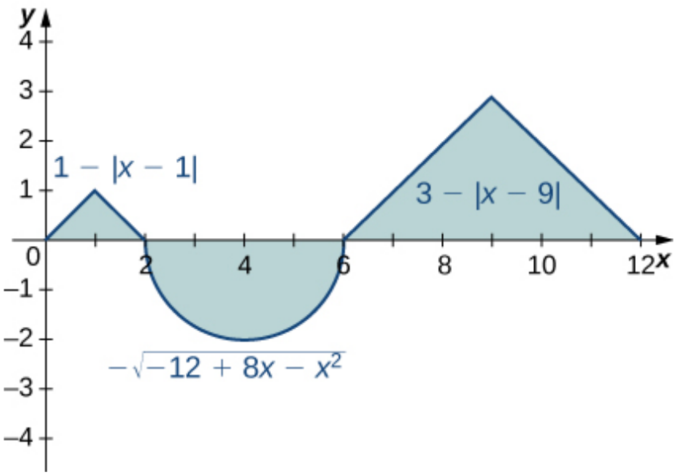
Solution: Draw the required rectangles:



Therefore the estimate for the integral is

$$L_4 = (1)(2) - (1)(1) - (1)(2) - (1)(1) = -2.$$

10. (8 points) Evaluate the integral of the functions graphed using formulas for the areas of triangles $\left(\frac{1}{2}bh\right)$ and circles (πr^2):



Solution: The first triangle has area $\frac{1}{2}(2)(1) = 1$, the semicircle has area $\frac{1}{2}\pi(2^2) = 2\pi$, and the second triangle has area $\frac{1}{2}(6)(3) = 9$. Therefore the integral is

$$\int_0^{12} f(x)\mathrm{d}x = \underbrace{\int_0^2 f(x)\mathrm{d}x}_{\text{first triangle}} + \underbrace{\int_2^6 f(x)\mathrm{d}x}_{\text{semicircle}} + \underbrace{\int_6^{12} f(x)\mathrm{d}x}_{\text{second triangle}} = 1 - 2\pi + 9 = 10 - 2\pi.$$