# MATH 2501 - EXAM 3 - FALL 2018
# SOLUTION

26 October 2018
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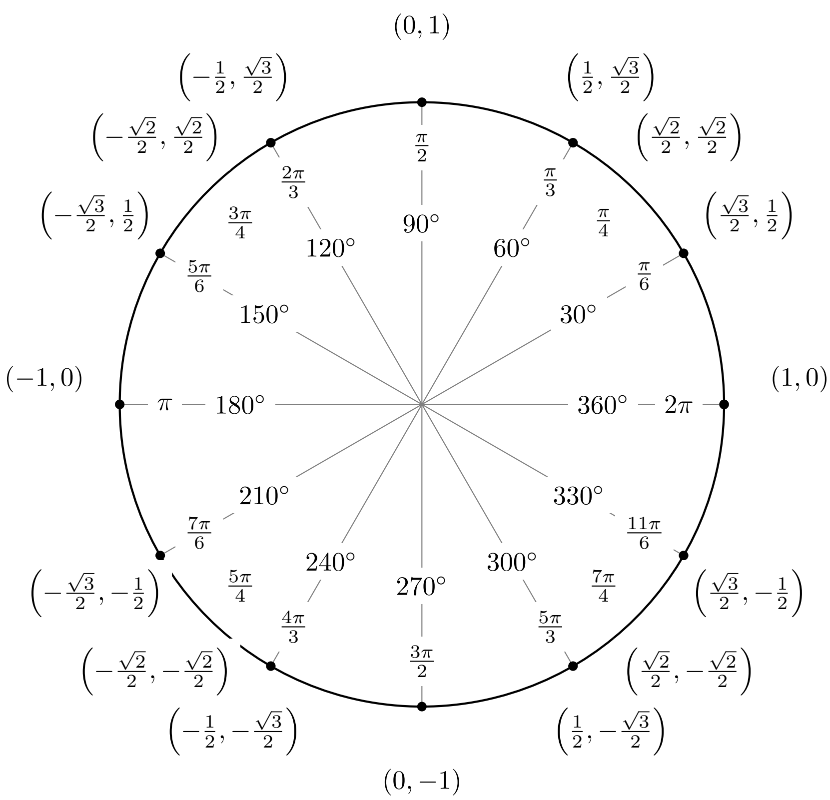
**Instructions:**

- Show all work, clearly and in order, if you want to get full credit. If you claim something is true **you must show work backing up your claim**. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers algebraically whenever possible to ensure full credit.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point.
- Good luck!

Formulas




**1st derivative test**:

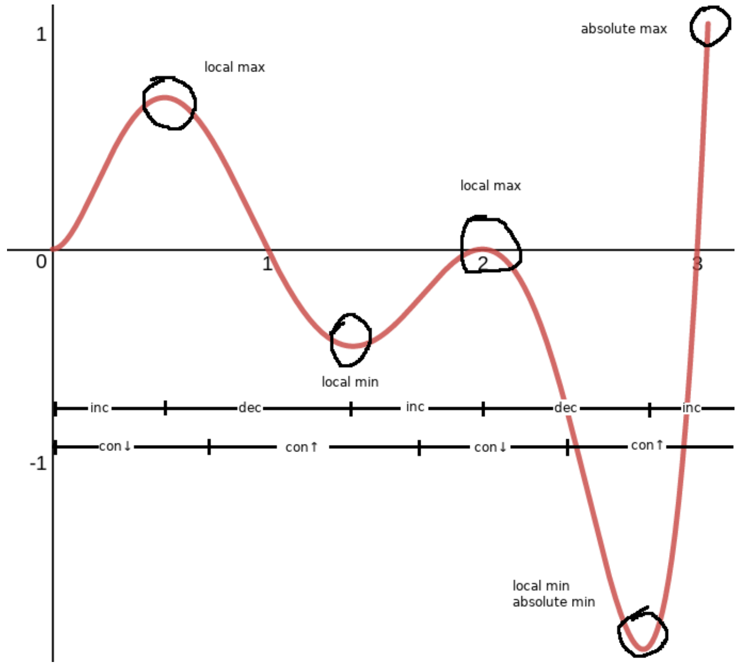
- if $f'$ changes sign from $\oplus$ to $\ominus$ at $c$, then there is a local maximum at $x = c$
- if $f'$ changes sign from $\ominus$ to $\oplus$ at $c$, then there is a local minimum at $x = c$

**2nd derivative test**: Let $c$ be a critical point of $f$.

- if $f''(c) > 0$, then $f$ has a local minimum at $x = c$
- if $f''(c) < 0$, then $f$ has a local maximum at $x = c$

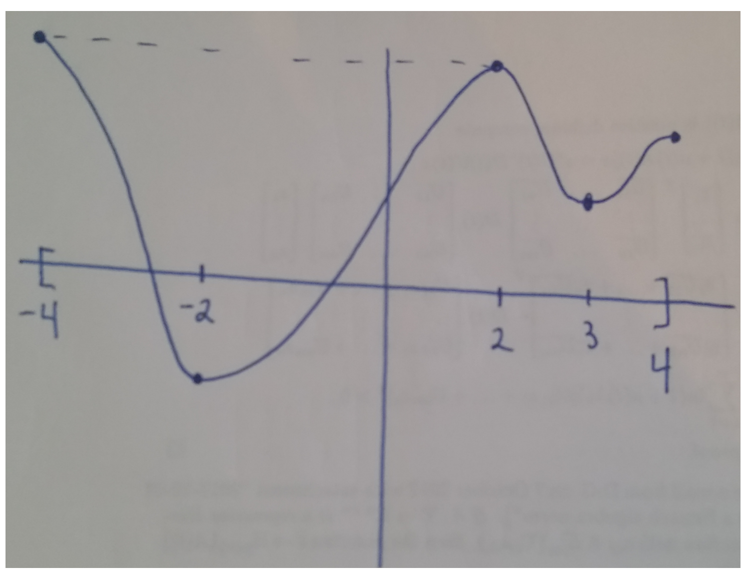
**Mean Value Theorem**: If $f$ is a continuous function on the interval $[a, b]$ that is differentiable on the interval $(a, b)$, then there is a point $c$ so that $a < c < b$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

1. (16 points) (a) (8 points) From the following graph, identify where the function is increasing and decreasing, any local minima and local maxima, and absolute extrema.




(b) (8 points) Draw a continuous function on the interval $[-4, 4]$ which has an absolute maximum at $x = -4$, a local minimum at $x = -2$, an absolute maximum at $x = 2$, and a local minimum at $x = 3$.

*Solution*:



2. (21 points) Consider the function $f(x) = \sin(x) - \cos(x)$ on the interval $[0, \pi]$.
   (a) (6 points) Compute $f'(x)$.
   *Solution*: $f'(x) = \cos(x) - (-\sin(x)) = \cos(x) + \sin(x)$
   (b) (7 points) Using your answer from (a), find the critical points of $f$.
   *Solution*: We must solve $f'(x) = 0$, i.e. $\cos(x) + \sin(x) = 0$. This means we seek angles $x$ for which $\cos(x) = -\sin(x)$, in other words, at the angles $x = \dfrac{3\pi}{4}$ and $x = \dfrac{7\pi}{4}$. However we are not considering $x = \dfrac{7\pi}{4}$ because $\dfrac{7\pi}{4}$ is not in the interval $[0, \pi]$.
   (c) (8 points) Using your answer to (b) and the given interval, find the absolute extrema of $f$. Express them in sentence form (i.e. "There is an absolute minimum of foo occurring at $x =$ bar.")
   *Solution*: Make a table:

| $x =$ | $f(x) =$ |
|---|---|
| $0$ | $\sin(0) - \cos(0) = 0 - 1 = -1$ |
| $\dfrac{3\pi}{4}$ | $\sin\left(\dfrac{3\pi}{4}\right) - \cos\left(\dfrac{3\pi}{4}\right) = \dfrac{\sqrt{2}}{2} - \left(-\dfrac{\sqrt{2}}{2}\right) = \sqrt{2}$ |
| $\pi$ | $\sin(\pi) - \cos(\pi) = 0 - (-1) = 1$ |

   From the table, we observe that an absolute maximum of $\sqrt{2}$ occurs at $x = \dfrac{3\pi}{4}$ and an absolute minimum of $-1$ occurs at $x = 0$.

3. (16 points) Consider the following scenario: at 10:00 AM you pass a police car going 55 miles per hour. You pass a second police car while going 55 miles per hour at 10:30 AM which is located 40 miles from the first police car. If the speed limit on the road is 75 miles per hour, then can the police cite you for speeding? If so, why? If not, then why not?
   *Solution*: Yes. The distance between the two cars is 40 miles and the time it took was 30 minutes, i.e., $\dfrac{1}{2}$ hour. This means that your position at time $t$ (measured in "hours since 10:00") is given by a function $d(t)$ where $d(0) = 0$ (position at the first car) and $d(\frac{1}{2}) = 40$ (position of second car). The average velocity of this function is

$$\text{AvgVel} = \frac{d(\frac{1}{2}) - d(0) \text{ miles}}{\frac{1}{2} - 0 \text{ hours}} = \frac{40}{\frac{1}{2}} \frac{\text{miles}}{\text{hour}} = 80 \frac{\text{miles}}{\text{hour}}.$$

   The mean value theorem guarantees that **there exists** a time $t = c$ for which $d'(c) = 80$ ("instantaneous rate of change of distance with respect to time"). In other words, **there exists** a time for which your speedometer reads $80 \dfrac{\text{miles}}{\text{hour}}$, i.e. you were speeding!

4. (15 points) In this problem, consider two positive real numbers $x$ and $y$ whose sum is 3.
   (a) (5 points) For such $x$ and $y$, express the quantity $Q = x^2 - 2xy$ in terms of only the variable $x$.
   *Solution*: To say "sum is 3" means that $x + y = 3$. Solving this for $y$ yields $y = 3 - x$. Now plugging this $y$ into $Q = x^2 - 2xy$ yields

$$Q = x^2 - 2x(3 - x) = x^2 - 6x + 2x^2 = 3x^2 - 6x.$$

   (b) (5 points) Write in the blanks below what quantities $x$ and $y$ minimize the expression for $Q$ you found in (a).
   *Solution*: To minimize, we have to find the critical point(s). First, compute

$$\frac{\mathrm{d}Q}{\mathrm{d}x} = 6x - 6.$$

Set this equal to zero to get $6x - 6 = 0$, hence $x = 1$. Plugging this into the formula $y = 3 - x$ yields $y = 2$.

$$x = \underline{\mathbf{1}} \qquad\qquad y = \underline{\mathbf{2}}$$

(c) (5 points) Explain why your answer in part (a) actually minimizes $Q$.
*Solution*: The first derivative test could be used here. However it is much easier to use the second derivative test:

$$\frac{\mathrm{d}^2 Q}{\mathrm{d}x^2} = 6$$

and so since $Q''(1) = 6 > 0$, the second derivative test guarantees that there is a local minimum at the critical point $x = 1$.

5. (12 points) Consider the function of two variables $f(x, y) = x^2 y^3 + \cos(x)\sin(y)$.

(a) (6 points) Compute the partial derivative $\dfrac{\partial f}{\partial x}$.
*Solution*: Compute

$$\frac{\partial f}{\partial x} = 2xy^3 - \sin(x)\sin(y)$$

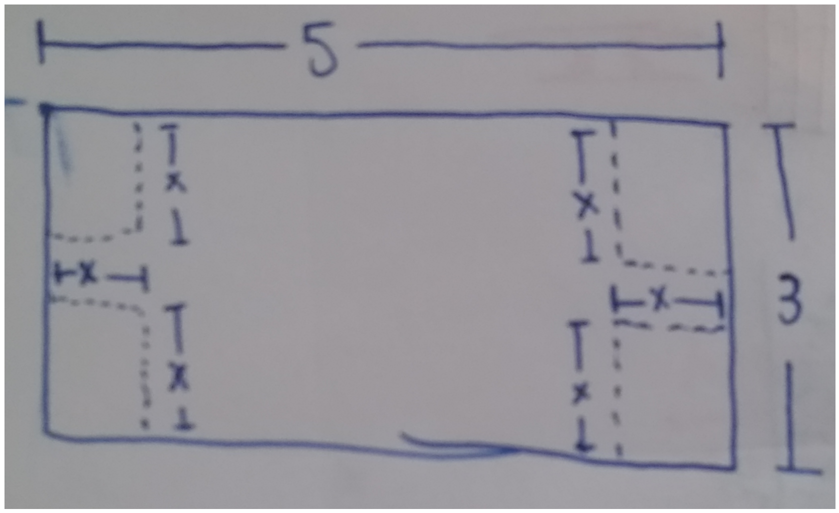
(b) (6 points) Compute the partial derivative $\dfrac{\partial f}{\partial y}$.
*Solution*: Compute

$$\frac{\partial f}{\partial y} = 3x^2 y^2 + \cos(x)\cos(y).$$

Do **ONE** of the following two problems: #6 or #7. Cross out the one you do NOT want graded.

6. (20 points) Suppose that you have a flat piece of cardboard that measures 3 meters by 5 meters. You want to cut equally sized squares of length $x$ from each corner of the flat piece of cardboard to fold up the edges, creating a box.

   (a) (5 points) Fill in the blanks to express what range of values $x$ may take due to the physical restrictions of the problem:
   *Solution*: First draw the scenario:




   Cutting $x = 0$ would give no height to the box, hence zero volume. The short side of the box restricts the cut to be $\leq 1.5$, but a cut of length exactly $x = 1.5$ would similarly give zero height to the box, hence zero volume. Therefore the physically meaningful bounds on the cut for this box are

$$\mathbf{0} < x < \mathbf{1.5}$$

   (b) (5 points) Write an equation for the volume $V$ of the box obtained when folding up the edges in terms of only the variable $x$.
   *Solution*: $V = \underbrace{(5-2x)}_{=\text{width}}\underbrace{(3-2x)}_{=\text{length}}\underbrace{x}_{=\text{height}}$
   Expanding this expression algebraically yields $V = 4x^3 - 16x^2 + 15x$.

   (c) (5 points) Find the critical point(s) of $V$.
   *Solution*: Compute

$$\frac{\mathrm{d}V}{\mathrm{d}x} = 12x^2 - 32x + 15 \overset{\text{SET}}{=} 0.$$

   Using the quadratic formula, we see that

$$x = \frac{-(-32) \pm \sqrt{(-32)^2 - 4(12)(15)}}{2(12)} = \frac{32 \pm \sqrt{1024 - 720}}{24} = \frac{32 \pm \sqrt{304}}{24}.$$

   We take the "$-$" solution since the "$+$" solution is too large (note that $\sqrt{304}$ is larger than 17 because $17^2 = 289$, hence the "$+$" solution is larger than $\dfrac{32+17}{24} > 2$, i.e. we throw it out).

(d) (5 points) Find the dimensions of the box with the largest volume and write them in the the three blanks below:
*Solution*:

Length: $3-2\left(\dfrac{32-\sqrt{304}}{24}\right)$ Width: $5-2\left(\dfrac{32-\sqrt{304}}{24}\right)$ Height: $\dfrac{32-\sqrt{304}}{24}$

7. (20 points) Consider a wire that is 5 feet long cut into two pieces. One piece forms a circle of radius $r$ and the other piece forms a square with side length $x$.

(a) (5 points) Fill in the blanks to express what range of values $x$ may take due to the physical restrictions of the problem:
*Solution*: Since the square has side length $x$, it has total perimeter $4x$. Therefore if $x=0$ then we are using all of the wire for the circle and if $4x=5$, i.e. $x=\frac{5}{4}$, then we are using all of the wire for the square:

$$0 \leq x \leq \frac{5}{4}$$

(b) (5 points) Write an equation for the area $A$ consisting of of the area of the square plus the area of the circle. Express $A$ in terms of only the variable $x$.
*Solution*: The radius of the circle $r$ is made of the length of the wire after $4x$ is removed. Therefore the radius $r = 5-4x$. Now we see

$$\text{Area} = \text{Area}_{\text{square}} + \text{Area}_{\text{circle}} = x^2 + \pi r^2 = x^2 + \pi(5-4x)^2.$$

(c) (5 points) Find the critical points of $A$.
*Solution*: Calculate

$$\frac{\mathrm{d}A}{\mathrm{d}x} = 2x + 2\pi(5-4x)(-4) \overset{\text{set}}{=} 0.$$

This yields

$$(2-8\pi)x + 10\pi = 0,$$

hence

$$x = \frac{-10\pi}{2-8\pi},$$

since $\pi \approx 3$, this quantity is approximately $\frac{-30}{2-24} = \frac{30}{22} = 1.36$, which is larger than $\frac{5}{4}$, so we will not consider it in the next part due to physical restriction.

(d) (5 points) Find the absolute extrema of $A$.
*Solution*: Make a table:

| $x=$ | $A(x)=$ |
|---|---|
| $0$ | $0+\pi(5-4(0))^2 = 25\pi \approx 75$ |
| $\frac{5}{4}$ | $\left(\frac{5}{4}\right)^2 = \frac{25}{16} + \pi(0) \approx 1.562$ |

so we see that the absolute maximum occurs when $x=0$ and the absolute minimum occurs when $x=\frac{5}{4}$.