MATH 2501 - EXAM 2 - FALL 2018
SOLUTION

4 October 2018
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Instructions:

- Show all work, clearly and in order, if you want to get full credit. If you claim something is true **you must show work backing up your claim**. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers algebraically whenever possible to ensure full credit.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point.
- Good luck!

1. (35 points) Compute the derivative of the function.

(a) (7 points) $f(t) = (t^2 + 5t)^{2018}$
Solution: Compute using the chain rule

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} f(t) &= \frac{\mathrm{d}}{\mathrm{d}t}(t^2 + 5t)^{2018} \\
&\overset{u=t^2+5t}{=} \frac{\mathrm{d}u}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}u} u^{2018} \\
&\overset{\frac{\mathrm{d}u}{\mathrm{d}t}=2t+5}{=} (2t+5)(2018)u^{2017} \\
&= 2018(2t+5)(t^2+5t)^{2017}.
\end{aligned}$$

(b) (7 points) $g(\theta) = \cos(\theta)\sin(\theta)$
Solution: Compute using the product rule

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\theta} g(\theta) &= \frac{\mathrm{d}}{\mathrm{d}\theta}\cos(\theta)\sin(\theta) \\
&= \left(\frac{\mathrm{d}}{\mathrm{d}\theta}\cos(\theta)\right)\sin(\theta) + \cos(\theta)\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\sin(\theta)\right) \\
&= -\sin(\theta)\sin(\theta) + \cos(\theta)\cos(\theta) \\
&= -\sin^2(\theta) + \cos^2(\theta).
\end{aligned}$$

(c) (7 points) $h(x) = e^{x^2-5x+2}$
Solution: Compute using the chain rule

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} h(x) &= \frac{\mathrm{d}}{\mathrm{d}x} e^{x^2-5x+2} \\
&\overset{u=x^2-5x+2}{=} \frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}u} e^u \\
&\overset{\frac{\mathrm{d}u}{\mathrm{d}x}=2x-5}{=} (2x-5)e^u \\
&= (2x-5)e^{x^2-5x+2}.
\end{aligned}$$

(d) (7 points) $\ell(w) = \ln(\ln(\ln(w)))$
Solution: Compute using the chain rule (2 times!)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}w}\ell(w) &= \frac{\mathrm{d}}{\mathrm{d}w}\ln(\ln(\ln(w))) \\
&\overset{u=\ln(\ln(w))}{=} \frac{\mathrm{d}u}{\mathrm{d}w}\frac{\mathrm{d}}{\mathrm{d}u}\ln(u) \\
&= \left[\frac{\mathrm{d}}{\mathrm{d}w}\ln(\ln(w))\right]\frac{1}{u} \\
&\overset{v=\ln(w)}{=} \left[\frac{\mathrm{d}v}{\mathrm{d}w}\frac{\mathrm{d}}{\mathrm{d}v}\ln(v)\right]\frac{1}{u} \\
&\overset{\frac{\mathrm{d}v}{\mathrm{d}w}=\frac{1}{w}}{=} \frac{1}{w}\frac{1}{v}\frac{1}{u} \\
&= \frac{1}{w}\frac{1}{\ln(w)}\frac{1}{\ln(\ln(w))}
\end{aligned}$$

(e) (7 points) $\alpha(\Omega) = \tan(\csc(\Omega))$

Solution: First note that

$$\frac{\mathrm{d}}{\mathrm{d}\Omega}\tan(\Omega) = \frac{\mathrm{d}}{\mathrm{d}\Omega}\frac{\sin(\Omega)}{\cos(\Omega)} \overset{\text{quotient rule}}{=} \frac{\cos^2(\Omega)+\sin^2(\Omega)}{\cos^2(\Omega)} \overset{\cos^2(\Omega)+\sin^2(\Omega)=1}{=} \frac{1}{\cos^2(\Omega)} = \sec^2(\Omega)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\Omega}\csc(\Omega) = \frac{\mathrm{d}}{\mathrm{d}\Omega}\frac{1}{\sin(\Omega)} \overset{\text{quotient rule}}{=} \frac{-1(\cos(\Omega))}{\sin^2(\Omega)} = -\cot(\Omega)\csc(\Omega).$$

Now compute using the chain rule

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\Omega}\alpha(\Omega) &= \frac{\mathrm{d}}{\mathrm{d}\Omega}\tan(\csc(\Omega)) \\
&\overset{u=\csc(\Omega)}{=} \frac{\mathrm{d}u}{\mathrm{d}\Omega}\frac{\mathrm{d}}{\mathrm{d}u}\tan(u) \\
&\overset{\frac{\mathrm{d}u}{\mathrm{d}\Omega}=-\cot(\Omega)\csc(\Omega)}{=} \Big(-\cot(\Omega)\csc(\Omega)\Big)\sec^2(u) \\
&= -\cot(\Omega)\csc(\Omega)\sec^2(\csc(\Omega))
\end{aligned}$$

2. (18 points) An isotope of the element iodine, iodine-131, has a half-life of approximately 8 days. Initially there are 6 grams of the isotope present. Consider the equation $M(t) = M_0 e^{kt}$, where $M(t)$ denotes the mass of the substance at time t, M_0 is the initial mass, k is the decay rate, and t is measured in days.

(a) (5 points) Write the exponential function that relates the amount of substance remaining as a function of t (note: this means to find k using the half-life property).

Solution: Since the half-life is 8 days, this means that after $t = 8$ days have passed, only half of the substance remains. Since we started with $M_0 = 6$ grams, this means in the time period of one half-life, there will be only $M = 3$ grams remaining. This yields the equation

$$3 = 6e^{8k}.$$

Divide by 6 to get

$$\frac{1}{2} = e^{8k}.$$

Take ln of both sides (exploiting the fact that the inverse function of e^x is $\ln(x)$) to arrive at

$$\ln\left(\frac{1}{2}\right) = 8k,$$

and finally, dividing by 8 yields

$$k = \frac{1}{8}\ln\left(\frac{1}{2}\right).$$

Therefore the exponential function that relates the amount of substance remaining as a function of t is

$$M(t) = 6e^{\frac{1}{8}\ln(\frac{1}{2})t}.$$

(b) (6 points) Using your answer to (a), determine the rate at which the substance is decaying in t days.

Solution: Calculate using the chain rule

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}M(t) &= \frac{\mathrm{d}}{\mathrm{d}t}6e^{\frac{1}{8}\ln(\frac{1}{2})t} \\
&= \frac{6}{8}\ln\left(\frac{1}{2}\right)e^{\frac{1}{8}\ln(\frac{1}{2})t}
\end{aligned}$$

(c) (7 points) Use your answer to (b) to determine the rate of decay at time $t = 10$ days.
Solution: Plugging in $t = 10$ into the equation we found above yields

$$\left.\frac{\mathrm{d}M}{\mathrm{d}t}\right|_{t=10} = \frac{6}{8}\ln\left(\frac{1}{2}\right)e^{\frac{10}{8}\ln\left(\frac{1}{2}\right)}\frac{\text{grams}}{\text{day}} \approx -0.218\frac{\text{grams}}{\text{day}}$$

3. (11 points) The circumference of a circle is increasing at a rate of $7\frac{\text{cm}}{\text{sec}}$. Find the rate of change of the radius in this situation.
Solution: The circumference C of a circle of radius r is given by the formula

$$C = 2\pi r.$$

Taking the derivative with respect to t of both sides yields

$$\frac{\mathrm{d}C}{\mathrm{d}t} = 2\pi\frac{\mathrm{d}r}{\mathrm{d}t}.$$

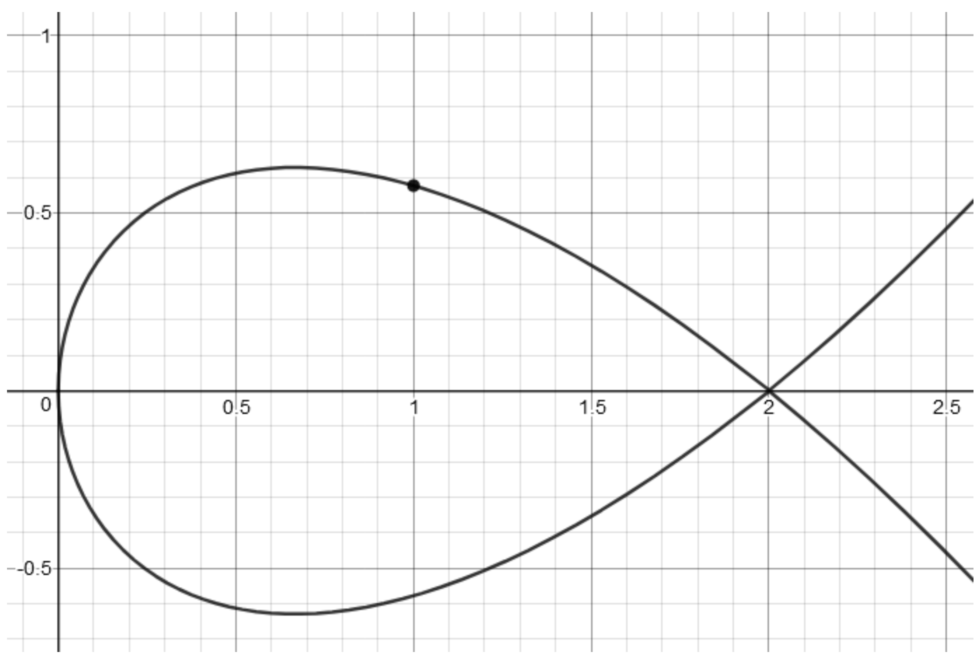
We are told that $\frac{\mathrm{d}C}{\mathrm{d}t} = 7$, so plug that in to arrive at

$$7 = 2\pi\frac{\mathrm{d}r}{\mathrm{d}t}.$$

Finally, divide by 2π to arrive at

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{7}{2\pi}\frac{\text{cm}}{\text{sec}} \approx 1.114\frac{\text{cm}}{\text{sec}}.$$

4. (16 points) The following is a plot of a so-called Tschirnhausen cubic:



which is given by the equation

$$3y^2 = x(x-2)^2.$$

(a) (8 points) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

Solution: Take $\dfrac{\mathrm{d}}{\mathrm{d}x}$ on both sides to get

$$6y\frac{\mathrm{d}y}{\mathrm{d}x} = (x-2)^2 + 2x(x-2).$$

Divide by $6y$ to get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x-2)^2 + 2x(x-2)}{6y}.$$

(b) (8 points) Using your answer to (a) above, find an equation of the tangent line to this curve at the point $(x_0, y_0) = \left(1, \dfrac{1}{\sqrt{3}}\right)$.

Solution: The slope is given by plugging $x = 1$ into $\dfrac{\mathrm{d}y}{\mathrm{d}x}$:

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=1, y=\frac{1}{\sqrt{3}}} = \frac{(1-2)^2 + 2(1)(1-2)}{6(\frac{1}{\sqrt{3}})} = \frac{\sqrt{3} - 2\sqrt{3}}{6}.$$

5. (20 points) "Ohm's law" says that the current through a conductor between two points is directly proportional to the voltage across the two points and obeys the equation

$$I = \frac{V}{R},$$

where I is the current through the conductor (in amps) , V is the voltage (in volts), and R is the resistance (in ohms).

If the current is increasing at a rate of $2\dfrac{\text{amps}}{\text{sec}}$ and voltage is decreasing at a rate of $3\dfrac{\text{volts}}{\text{sec}}$, then how fast is resistance changing when $I = 1$, $R = 3$, and $V = 3$?

Solution: There were two common approaches that are equivalent, but appear slightly different. I will write out both below:

Way 1: Just take $\dfrac{\mathrm{d}}{\mathrm{d}t}$ of both sides (using quotient rule on the right) to get

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{R\frac{\mathrm{d}V}{\mathrm{d}t} - V\frac{\mathrm{d}R}{\mathrm{d}t}}{R^2}.$$

From here, solve for $\dfrac{\mathrm{d}R}{\mathrm{d}t}$ to get

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \frac{R^2\dfrac{\mathrm{d}I}{\mathrm{d}t} - R\dfrac{\mathrm{d}V}{\mathrm{d}t}}{-V}.$$

From the written, we are told that $\dfrac{\mathrm{d}I}{\mathrm{d}t} = 2$ and $\dfrac{\mathrm{d}V}{\mathrm{d}t} = -3$. Taking these and $I = 1, R = 3$, and $V = 3$ in $\dfrac{\mathrm{d}I}{\mathrm{d}t}$ yields

$$\begin{aligned}\frac{\mathrm{d}R}{\mathrm{d}t} &= \frac{3^2(2) - 3(3)}{-(-3)} \\ &= \frac{18-9}{3} \\ &= \frac{9}{3} \\ &= 3.\end{aligned}$$

Way 2: Another approach is to solve the equation $I = \dfrac{V}{R}$ for R to get $R = \dfrac{V}{I}$. Now differentiate with respect to t (using the quotient rule) to get

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \frac{I\frac{\mathrm{d}V}{\mathrm{d}t} - V\frac{\mathrm{d}I}{\mathrm{d}t}}{I^2}.$$

Now plugging in the given data yields

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \frac{1(-3) - (-3)(2)}{1^2} = 3.$$

6. (3 points) **(Bonus)** Notice that

$$\begin{aligned} 1 &= 1^2 \\ 2 + 2 &= 2^2 \\ 3 + 3 + 3 &= 3^2 \\ 4 + 4 + 4 + 4 &= 4^2 \\ &\vdots \end{aligned}$$

Therefore

$$(*) \qquad \underbrace{x + x + x + \ldots + x}_{x \text{ terms}} = x^2$$

Explain what's wrong with the following argument: differentiating $(*)$ with respect to x yields

$$\underbrace{1 + 1 + 1 + \ldots + 1}_{x \text{ terms}} = 2x$$

which simplifies to

$$x = 2x,$$

and upon dividing by x yields

$$1 = 2.$$

Solution: The number of terms depends on x, and so the derivative does not work in the expected way.