MATH 2501 - EXAM 1 - FALL 2018
SOLUTION

7 September 2018
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Instructions:

- Show all work, clearly and in order, if you want to get full credit. If you claim something is true **you must show work backing up your claim**. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers algebraically whenever possible to ensure full credit.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point.
- Good luck!

1. (15 points) For the function $f(x) = 3 + \sqrt{x+7}$ find...

(a) (3 points) Find the domain.
Solution: The domain cannot allow values of x that make $x+7$ negative. In other words, we must avoid x-values that cause $x + 7 < 0$. Solving this inequality for x yields $x < -7$ – these points are to be avoided. Therefore the domain is $[-7, \infty)$.

(b) (3 points) Find the range.
Solution: Since the standard function $y = \sqrt{x}$ has range $[0, \infty)$ and this function has a vertical shift of 3 up, we observe that the range of $3 + \sqrt{x+7}$ must be $[3, \infty)$.

(c) (3 points) Find any x-intercepts (if any exist).
Solution: x-intercepts are places where the graph crosses the x-axis, i.e. we want to find solutions of the equation

$$3 + \sqrt{x+7} = 0.$$

Substract 3 to get

$$\sqrt{x+7} = -3.$$

Now square both sides to remove the square root to get

$$x + 7 = 9,$$

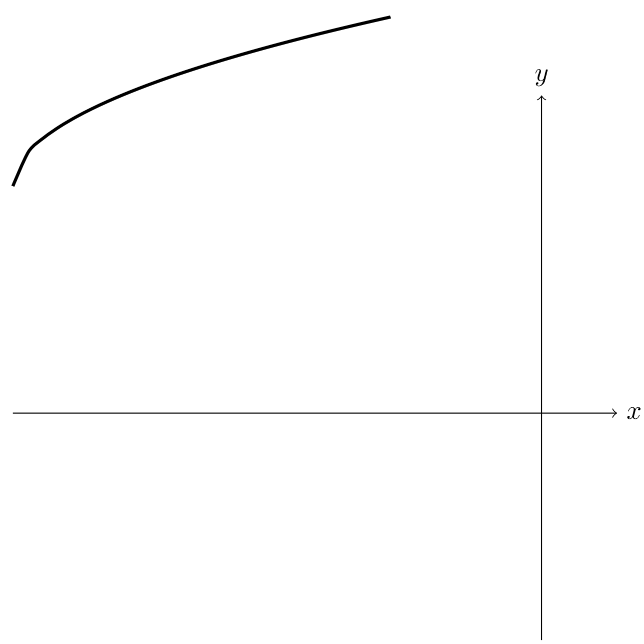
and finally subtract 7 to get $x = 2$. But is this a solution? It turns out no, because if we substitute $x = 2$ into $f(x)$ we get $f(2) = 3 + \sqrt{2+7} = 3 + \sqrt{9} = 3 + 3 = 6$.
Therefore there are no x-intercepts. (*note: how can this be if we solved the equation? Recall from college algebra that squaring an equation may introduce "extraneous roots", i.e. "solutions" obtained via moving symbols around, but aren't actually solutons. That is what happened here.*)

(d) (3 points) Find any y-intercepts (if any exist).
Solution: y-intercepts are places where the graph crosses the y-axis – to find them simply plug $x = 0$ into the function:

$$f(0) = 3 + \sqrt{0+7} = 3 + \sqrt{7}.$$

Therefore the y-intercept is $(0, 3 + \sqrt{7})$.

(e) (3 points) Sketch the function f.
Solution: Plot by applying a left shift by 7 and a vertical shift up by 3 to the basic function $y = \sqrt{x}$ to get



2. (10 points) A bacterial colony grown in a lab is known to double in number in 10 hours. Suppose, initially, there are $B_0 = 500$ bacteria present.

   (a) (5 points) Use the formula $B = B_0 e^{kt}$ to determine the value for k.
   Solution: The given information tells us that the original population of $B_0 = 500$ doubles to $B = 1000$ at time $t = 10$ to give us the equation

   $$1000 = 500e^{10k}.$$

   To solve it, first divide by 500 to get

   $$2 = e^{10k}.$$

   Now take the natural logarithm of both sides to get

   $$\ln(2) = \ln(e^{10k}).$$

   Since $\ln(x)$ and e^x are inverse functions, this gives us

   $$\ln(2) = 10k.$$

   Therefore find k by dividing by 10:

   $$k = \frac{\ln(2)}{10}.$$

   (b) (5 points) Determine how long it takes for 100,000 bacteria to grow.
   Solution: In this case, we will plug in our value of k we found in part (a) and set $B = 100000$ to get the equation

   $$100000 = 500e^{\frac{\ln(2)}{10}t}.$$

   To solve this for the time, first divide by 500 to get

   $$\frac{100000}{500} = e^{\frac{\ln(2)}{10}t}.$$

   Simplifying the left-hand side yields

   $$200 = e^{\frac{\ln(2)}{10}t}.$$

   Now plug both sides into $\ln(x)$ to get

   $$\ln(200) = \frac{\ln(2)}{10}t.$$

   Finally solve for t to get

   $$\frac{10\ln(200)}{\ln(2)} = t.$$

3. (7 points) Draw a function that has the following properties:

- the domain of the function is all real numbers
- $\lim_{x\to 1} f(x) = 4$
- $f(1) = 3$
- $\lim_{x\to 3^-} f(x) = 2, \ \lim_{x\to 3^+} = -\infty$
- $f(3) = 5$
- $\lim_{x\to -\infty} f(x) = \infty, \ \lim_{x\to\infty} f(x) = -\infty$

4. (13 points) Compute the following limits:

(a) (5 points) $\lim_{x\to 2} x^2 + 3x$

Solution: In this case we may just plug in $x = 2$ to arrive at

$$\lim_{x\to 2} x^2 + 3x = 2^2 + 3(2) = 4 + 6 = 10.$$

(b) (8 points) $\lim_{x\to 3} \frac{\sqrt{x+6}-3}{x-3}$

Solution: If we plug in $x = 3$, it results in indeterminant form $\frac{0}{0}$, thus there is more work to do. So compute

$$\begin{aligned}
\frac{\sqrt{x+6}-3}{x-3} &= \left(\frac{\sqrt{x+6}-3}{x-3}\right)\left(\frac{\sqrt{x+6}+3}{\sqrt{x+6}+3}\right) \\
&= \frac{x+6-9}{(x-3)(\sqrt{x+6}+3)} \\
&= \frac{x-3}{(x-3)(\sqrt{x+6}+3)} \\
&= \frac{1}{\sqrt{x+6}+3}.
\end{aligned}$$

Therefore we now compute

$$\begin{aligned}
\lim_{x\to 3} \frac{\sqrt{x+6}-3}{x-3} &= \lim_{x\to 3} \frac{1}{\sqrt{x+6}+3} \\
&\overset{\text{plug in } x=3}{=} \frac{1}{\sqrt{3+6}+3} \\
&= \frac{1}{\sqrt{9}+3} \\
&= \frac{1}{6}.
\end{aligned}$$

5. (11 points) Electric power P is related to the voltage difference V and current I by the formula

$$P = VI.$$

(a) (3 points) Solve the formula for voltage.
Solution: To solve for V, divide both sides by I to get

$$\frac{P}{I} = V.$$

(b) (5 points) Using your answer to (a), if the power remains at a constant value $P = 2$, then what happens to voltage in the limit as $I \to 0^+$?
Solution:

$$\lim_{I \to 0^+} \frac{2}{I} = +\infty$$

(c) (3 points) Explain what your answer to (b) means physically.
Solution: As current approaches zero, if power is to remain constant, then the voltage needs to approach infinity.

6. (10 points) Consider the function $h(x)$ given by

$$h(x) = \begin{cases} x^2, & x \leq 5 \\ x + k, & x > 5. \end{cases}$$

Find the value of k that makes the function continuous.
Solution: Notice that the place where continuity is questionable is only at $x = 5$. We need to force the following two limits to be equal:

$$\lim_{x \to 5^-} h(x) \overset{\text{compute}}{=} 5^2 = 25$$

and

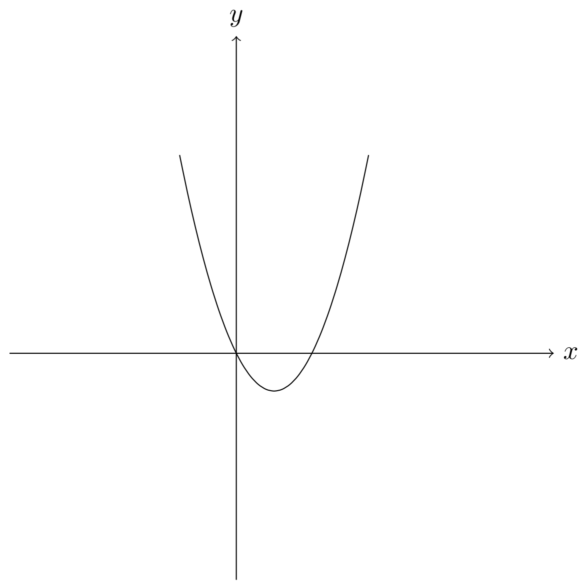
$$\lim_{x \to 5^+} h(x) \overset{\text{compute}}{=} 5 + k.$$

Therefore we have to solve the following equation for k:

$$25 = 5 + k.$$

Thus $k = 25 - 5 = 20$.

7. (10 points) Draw a possible picture of the derivative of the given function:



8. (10 points) The equation $2^x = x^2 + 1$ has zeros at $x = 0$ and $x = 1$. Explain why there must be a third zero on the interval $[3, 5]$.
Solution: Subtract $x^2 + 1$ to get the equation

$$2^x - x^2 - 1 = 0.$$

Call the function of x on the left-hand side $f(x)$, i.e.

$$f(x) = 2^x - x^2 - 1.$$

Note that this function **is continuous**. To finish, we need to argue that $f(x)$ has a zero in this interval. To do it, first compute

$$f(3) = 2^3 - 3^2 - 1 = 8 - 9 - 1 = -2$$

and

$$f(5) = 2^5 - 5^2 - 1 = 32 - 25 - 1 = 6.$$

Therefore, taking $z = 0$ we observe that

$$f(3) \leq z = 0 \leq f(5).$$

So, since f is continuous and $f(3) \leq 0 \leq f(5)$, we may apply the intermediate value theorem to say that there exists a number c so that $3 \leq c \leq 5$ and that $f(c) = 0$.

9. (14 points) Use the **definition** of the derivative to compute

(a) (6 points) $f'(x)$ when $f(x) = 5x + 2$.

Solution: Calculate

$$\begin{aligned} f'(x) &\overset{\text{definition}}{=} \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{[5(x+h)+2] - [5x+2]}{h} \\ &= \lim_{h\to 0} \frac{5x + 5h + 2 - 5x - 2}{h} \\ &= \lim_{h\to 0} \frac{5h}{h} \\ &= \lim_{h\to 0} 5 \\ &= 5. \end{aligned}$$

(b) (8 points) $f'(1)$ when $f(x) = \dfrac{1}{x-2}$.

Solution: Calculate

$$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{1+h-2} - \frac{1}{1-2}}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{h-1} - (-1)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1+h-1}{h-1}}{h} \\ &= \lim_{h\to 0} \frac{h}{h(h-1)} \\ &= \lim_{h\to 0} \frac{1}{h-1} \\ &= -1. \end{aligned}$$

Formulas

A function f is called continuous at $x = a$ provided that

i.) $f(a)$ exists,

ii.) $\lim_{x \to a} f(x)$ exists, and

iii.) $\lim_{x \to a} f(x) = f(a)$.

The derivative of a function f at a value $x = a$ is computed by

$$\begin{aligned} f'(a) &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \\ &\overset{\text{OR}}{=} \lim_{b \to a} \frac{f(b) - f(a)}{b - a} \end{aligned}$$

Intermediate Value Theorem: If $f(x)$ is continuous on the interval $[a, b]$ and z is a number so that $f(a) \le z \le f(b)$, then there is a domain value $x = c$ so that $f(c) = z$.

FACT: $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$