Quiz 2 — MATH 1586 Spring 2018

1. Recall Newton's law of cooling:

$$\frac{\mathrm{d}T}{\mathrm{d}t} = k(T - T_0),$$

where T denotes the temperature at time t, k is a proportionality constant, and T_0 is the ambient temperature of the surrounding space.

When a cake is removed from an oven, its temperature is $280°F$. 7 minutes later, it is $210°F$. How long will it take for the cake to cool down to $80°F$ in a room that is $74°F$?
Solution: We are told that $T_0 = 74$, so the differential equation becomes

$$\frac{\mathrm{d}T}{\mathrm{d}t} = k(T - 74).$$

Separate variables and integrate to get

$$\int \frac{1}{T-74}\mathrm{d}T = \int k\mathrm{d}t.$$

Calculate the integrals to get

$$\ln(T - 74) = kt + C,$$

and plug both sides into e^x to get

$$T - 74 = e^{kt+C} = C_1 e^{kt}, \quad C_1 = e^C,$$

hence our model is

$$(*) \qquad T = 74 + C_1 e^{kt}.$$

We are told two initial conditions: the temperature of the cake leaving the oven yields $T(0) = 280$ and the condition 7 minutes later yields $T(7) = 210$. Use the first condition to see

$$\underbrace{280}_{\text{given}} = T(0) = \underbrace{74 + C_1 e^0}_{\text{calculated}}.$$

Therefore we must solve $280 = 74 + C_1$ for C_1, yielding

$$C_1 = 280 - 74 = 206.$$

Therefore we may update our model to get

$$(**) \qquad T = 74 + 206e^{kt}.$$

Now applying the second condition yields

$$\underbrace{210}_{\text{given}} = T(7) = \underbrace{74 + 206e^{7k}}_{\text{calculated}}.$$

Therefore we must solve $210 = 74 + 206e^{7k}$ for k. This means

$$k = \frac{1}{7}\ln\left(\frac{210-74}{206}\right) = \frac{1}{7}\ln\left(\frac{136}{206}\right).$$

So we may update our model again to get

$$(***) \qquad\qquad T = 74 + 206e^{\frac{1}{7}\ln\left(\frac{136}{206}\right)t}.$$

Finally, to answer the question of at what time the cake is at temperature $80°F$, we see that we must solve

$$\underbrace{80}_{\text{given}} = T(t) = \underbrace{74 + 206e^{\frac{t}{7}\ln\left(\frac{136}{206}\right)}}_{\text{calculated}}.$$

This implies

$$6 = 206e^{\frac{t}{7}\ln\left(\frac{136}{206}\right)},$$

so

$$\ln\left(\frac{6}{206}\right) = \frac{t}{7}\ln\left(\frac{136}{206}\right),$$

therefore

$$t = \frac{7\ln\left(\frac{6}{206}\right)}{\ln\left(\frac{136}{206}\right)} \approx 59.61.$$

2. Compute

$$\int xe^{3x}\mathrm{d}x.$$

Solution: Let $u = x$ and $\mathrm{d}v = e^{3x}$. Then $\mathrm{d}u = \mathrm{d}x$ and $v = \frac{1}{3}e^{3x}$. Therefore using integration by parts we see

$$\int xe^{3x}\mathrm{d}x = \frac{x}{3}e^{3x} - \frac{1}{3}\int e^{3x}\mathrm{d}x = \frac{x}{3}e^{3x} - \frac{1}{9}e^{3x} + C.$$

3. Compute

$$\int_3^\infty e^{-5x}\mathrm{d}x.$$

Solution: Compute

$$\begin{aligned}\int_3^\infty e^{-5x}\mathrm{d}x &= \lim_{b\to\infty}\int_3^b e^{-5x}\mathrm{d}x \\ &= \lim_{b\to\infty} -\frac{1}{5}e^{-5x}\Big|_3^b \\ &= \lim_{b\to\infty} -\frac{e^{-5b}}{5} + \frac{1}{5}e^{-15} \\ &= \frac{1}{5}e^{-15}.\end{aligned}$$