("Simple chemical conversion") From chemical experiments, it is known that for certain reactions where a substance converts into another substance, the rate of change (with respect to time) of the amount x at any time $t > 0$ is determined by the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -kx(t),$$

for some constant k (we use "$-k$" in the equation because the amount of the substance is decreasing as it converts). The initial amount of the substance at time $t = 0$ is $x(0) = x_0$.

Suppose also that it is known (by measurement) that half of the substance converts by the end of 10 seconds.

At what time does $\frac{9}{10}$ of the substance become converted?

Solution: Solve the differential equation by "separation of variables", i.e. "illegally" splitting up the $\frac{\mathrm{d}x}{\mathrm{d}t}$ by "multiplying" by $\mathrm{d}x$ to get

$$\mathrm{d}x = -kx\mathrm{d}t.$$

Divide both sides of this equation by x (so that the x is on the same side as $\mathrm{d}x$) to get

$$\frac{1}{x}\mathrm{d}x = -k\mathrm{d}t.$$

(Illegally) integrate both sides by placing a $\int$ symbol in the front of each side:

$$\int \frac{1}{x}\mathrm{d}x = \int -k\mathrm{d}t.$$

Integrate on both sides (recall that the constant $-k$ can pull outside of the integral) to get

$$\ln(x) = -kt + C.$$

To solve for the function x, plug both sides into the exponential function to get the function

$$x(t) = e^{-kt+C} = e^C e^{-kt} = C_1 e^{-kt}, \quad C_1 = e^C.$$

Find C_1

Using the initial condition $x(0) = x_0$, we observe that

$$\underbrace{x_0}_{\text{given}} = x(0) = \underbrace{C_1 e^{-k0} = C_1 e^0 = C_1}_{\text{calculated}}$$

Therefore $C_1 = x_0$. This means our model is

$$x(t) = x_0 e^{-kt}.$$

Find k

To find k we use the statement that half of the substance is gone at time $t = 10$ to write

$$\underbrace{\frac{1}{2}x_0}_{\text{given}} = x(10) = \underbrace{x_0 e^{-10k}}_{\text{calculated}}.$$

Now solve this for k: divide both sides by x_0 to get

$$\frac{1}{2} = e^{-10k}.$$

Take ln of both sides to get

$$\ln\left(\frac{1}{2}\right) = -10k,$$

and finally divide by -10 to get

$$k = -\frac{1}{10}\ln\left(\frac{1}{2}\right) = \frac{\ln(2)}{10} = 0.0693147\ldots.$$

Therefore our model is

$$x(t) = x_0 e^{-\frac{\ln(2)}{10}t}.$$

To answer the question "At what time does $\frac{9}{10}$ of the substance become converted?" we must be careful in setting up the equation. If $\frac{9}{10}$ of the subtsance has converted, then how much is left? The answer is $\frac{1}{10}$! Therefore we want to solve for t in the following equation:

$$\frac{1}{10}x_0 = x(t) = x_0 e^{-\frac{\ln(2)}{10}t}.$$

Dividing by x_0 yields

$$\frac{1}{10} = x(t) = e^{-\frac{\ln(2)}{10}t}.$$

Taking ln of both sides yields

$$\ln\left(\frac{1}{10}\right) = -\frac{\ln(2)}{10}t.$$

Hence

$$t = -\frac{10}{\ln(2)}\ln\left(\frac{1}{10}\right) = \frac{10\ln(10)}{\ln(2)} = 33.219280\ldots.$$