Homework 7 — MATH 1586 Spring 2018

Recall that the tangent plane to a surface $z = f(x, y)$ above the point (a, b) is given by

$$z = f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b).$$

Recall that the gradient of a function $f(x, y)$ or a function $f(x, y, z)$ is given by

$$\nabla f(x, y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

Recall the magnitude of a vector $\langle a, b\rangle$ is its length, i.e. $\|\langle a, b\rangle\| = \sqrt{a^2 + b^2}$. We say that a vector is a unit vector if it is length 1. You can make any vector $\vec{v}$ into a unit vector $\vec{u}$ pointing in the same direction as $\vec{v}$ by dividing by its magnitude: $\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|}$.

Let $\vec{u}$ be a unit vector and let $f(x, y)$ be a function. We say that the directional derivative of f in the direction $\vec{u} = \langle a, b\rangle$ at (x_0, y_0) is given by

$$D_{\vec{u}} f(x_0, y_0) = \vec{u} \cdot \nabla f(x_0, y_0) = a f_x(x_0, y_0) + b f_y(x_0, y_0).$$

Also recall the following theorem:

Theorem: Suppose that $f(\vec{x})$ where $\vec{x} = \langle x_1, \ldots, x_n\rangle$ is a partial differentiable function of all variables $x_1, x_2, \ldots, x_n$. The maximum value of $D_u f(\vec{x})$ is $\|\nabla f(\vec{x})\|$ and it occurs in the same direction as the gradient vector $\nabla f(\vec{x})$. ■

Recall that to compute "iterated integrals" like

$$\int_a^b \int_c^d f(x, y) \mathrm{d}x \mathrm{d}y$$

is just to work "inside-out". The region of integration of such an integral is the square whose top and bottom sides are above $[a, b]$ on the x-axis and whose left and right sides are next to $[c, d]$ on the y-axis.

1. Find an equation for the tangent plane to $f(x, y) = x^2y^3 + xy^2$ above the point $(2, 3)$.
   Solution: First find
   $$\frac{\partial f}{\partial x} = 2xy^3 + y^2$$
   and
   $$\frac{\partial f}{\partial y} = 3x^2y^2 + 2xy.$$
   Therefore the tangent plane is given by
   $$z = f(2, 3) + f_x(2, 3)(x - 2) + f_y(2, 3)(y - 3).$$
   Compute
   $$f(2, 3) = 2^2 3^3 + 2(3^2) = 126,$$

$$f_x(2,3) = 2(2)(3^3) + 3^2 = 117,$$

and

$$f_y(2,3) = 3(2^2)(3^2) + 2(2)(3) = 120.$$

Therefore the tangent plane is given by the equation

$$z = 126 + 117(x-2) + 120(y-3).$$

2. Calculate the gradient of $f(x,y) = x^2 e^{xy} + y\log(x)$.
   Solution: Calculate

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \left\langle 2xe^{xy} + x^2 y e^{xy} + \frac{y}{x}, x^3 e^{xy} + \log(x) \right\rangle.$$

3. Find a unit vector that points in the same direction as the vector $\vec{v} = \langle 2, 3 \rangle$.
   Solution:

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\vec{v}}{\sqrt{2^2+3^2}} = \frac{\langle 2,3 \rangle}{\sqrt{13}} = \left\langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right\rangle.$$

4. Find the directional derivative of $f(x,y) = xy^2 + e^{xy}$ in the direction $\vec{u} = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$ (note: this vector is already a unit vector) at $(x_0, y_0) = (1,2)$.
   Solution: First find the gradient of f:

$$\nabla f = \left\langle y^2 + ye^{xy}, 2xy + xe^{xy} \right\rangle.$$

   Now the directional derivative is

$$\begin{aligned} D_{\vec{u}}(1,2) = \vec{u} \cdot \nabla f(1,2) &= \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle \cdot \left\langle 4 + 2e^2, 4 + e^2 \right\rangle \\ &= \frac{\sqrt{2}}{2}(4 + 2e^2 + 4 + e^2) = \frac{\sqrt{2}}{2}(8 + 3e^2) \end{aligned}$$

5. Find the directional derivative of $f(x,y) = e^x \log(y) + x^2 y$ in the direction of $\langle -1, 2 \rangle$ at $(x_0, y_0) = (2,2)$.
   Solution: The vector $\langle -1, 2 \rangle$ is not a unit vector, so normalize it:

$$\vec{u} = \frac{\langle -1,2 \rangle}{\|\langle -1,2 \rangle\|} = \frac{\langle -1,2 \rangle}{\sqrt{5}} = \left\langle \frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$$

   Find the gradient of f:

$$\nabla f = \left\langle e^x \log(y) + 2xy, \frac{e^x}{y} + x^2 \right\rangle$$

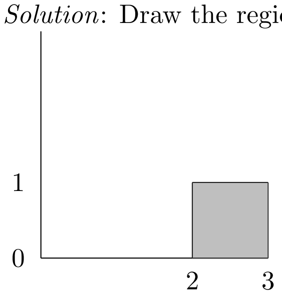
Now calculate the directional derivative:

$$D_{\vec{u}}f(2,2) = \nabla f(2,2)\cdot\vec{u} = \left\langle e^2\log(2)+8, \frac{e^2}{2}+4\right\rangle \cdot \left\langle \frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right\rangle$$
$$= -\frac{e^2\log(2)+8}{\sqrt{5}} + \frac{2(\frac{e^2}{2}+4)}{\sqrt{5}}$$

6. Calculate

$$\int_0^1\int_2^3 x^2y + xy^2 \mathrm{d}x\mathrm{d}y$$

and draw the region of integration.
Solution: Draw the region of integration:



Now calculate

$$\begin{aligned}\int_0^1\int_2^3 x^2y + xy^2 \mathrm{d}x\mathrm{d}y &= \int_0^1 \frac{x^3y}{3} + \frac{x^2y^2}{2}\Big|_{x=2}^{x=3}\mathrm{d}y\\ &= \int_0^3 \left(\frac{27y}{3}+\frac{9y^2}{2}\right) - \left(\frac{8y}{3}+\frac{4y^2}{2}\right)\mathrm{d}y\\ &= \int_0^3 \frac{19y}{3}+\frac{5y^2}{2}\mathrm{d}y\\ &= \frac{19y^2}{3}+\frac{5y^2}{6}\Big|_0^3\\ &= \frac{19(9)}{3}+\frac{5(9)}{6}\\ &= 57+10\\ &= 67.\end{aligned}$$