Homework 4 — MATH 1586 Spring 2018

Recall the geometric series: for $|r| < 1$, we have $\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$. If $|r| > 1$, then $\sum_{k=0}^{\infty} r^k$ diverges.

The ratio test: to see if $\sum_{k=0}^{\infty} a_k$ converges or not, compute $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|$. If that limit is > 1, the series diverges. If that limit is < 1, then the series converges. If that limit equals 1, then the ratio test offers no conclusion.
The Taylor series (centered at 0) of a function $f(x)$ is given by

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k,$$

where $f^{(k)}$ denotes the kth derivative of f (with $k = 0$ meaning take no derivative).

1. Calculate $\sum_{k=1}^{4} \frac{1}{k^2 + 2k + 1}$

2. Does $\sum_{k=0}^{\infty} 2^k$ converge? If so, find its value. If it does not, why not?

3. Does $\sum_{k=0}^{\infty} \frac{1}{(-3)^k}$ converge? If so, find its value. If it does not, why not?

4. The sine function sin is defined by the series

$$\sin(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{(2k+1)!}.$$

   The so-called "sine integral function" Si (used in, e.g. signal processing) is defined by the formula

$$\text{Si}(x) = \int_0^x \frac{\sin(t)}{t} \mathrm{d}t.$$

   Use the Taylor series for $\sin(x)$ and integration term-by-term to find the Taylor series for $\text{Si}(x)$.

5. Use the ratio test to try to conclude whether or not the series $\sum_{k=0}^{\infty} \frac{1}{k+1}$ converges.

6. Use the ratio test to try to conclude whether or not the series $\sum_{k=0}^{\infty} \frac{1}{k2^k}$ converges.

7. Use the ratio test to try to conclude whether or not the series $\sum_{k=0}^{\infty} \frac{k!}{2^k}$ converges.

8. Use the definition of Taylor series to compute the first three nonzero terms of the Taylor series for $f(x) = e^x \ln(x^2 + 1)$.