Homework 3 — MATH 1586 Spring 2018

Recall the technique of integration by parts is of the form

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u.$$

Also recall that we derived the antiderivative of the natural logarithm using integration by parts:

$$\int \ln(x)\mathrm{d}x = x\ln(x) - x + C.$$

Recall that an "improper integral" of the form $\displaystyle\int_a^\infty$ or of the form $\displaystyle\int_{-\infty}^b$ is understood in the following way:

$$\int_a^\infty f(x)\mathrm{d}x = \lim_{b\to\infty}\int_a^b f(x)\mathrm{d}x.$$

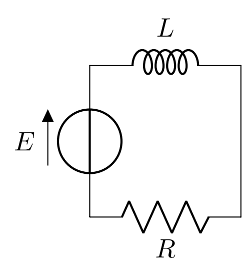
and

$$\int_{-\infty}^b f(x)\mathrm{d}x = \lim_{a\to-\infty}\int_a^b f(x)\mathrm{d}x.$$

Finally, recall the definition of the Laplace transform of a function f:

$$\mathscr{L}\{f\}(x) = \int_0^\infty f(t)e^{-xt}\mathrm{d}t.$$

1. ("RL-Circuit problem")



In a series circuit containing only a resistor (R, measured in ohms), an inductor (L, measured in henries which are ohm $\cdot$ sec), and a current at time t, $i(t)$ (measured in amps), Kirchoff's second law states that the sum of the voltage drop across the inductor $\left(L\frac{\mathrm{d}i}{\mathrm{d}t}\right)$ and the voltage drop across the resistor (iR) is the same as the impressed voltage ($E(t)$) on the circuit. We obtain the differential equation for the current $i(t)$

$$L\frac{\mathrm{d}i}{\mathrm{d}t} + Ri(t) = E(t),$$

where L and R are constants known as the inductance and resistance.

A 25-volt electromotive force is applied to a series circuit in which the inductance is 1 henry and the resistance is 30 ohms. $i' + 30i = 25$

a.) What is the differential equation we must solve here?
Solution: $\frac{\mathrm{d}i}{\mathrm{d}t} + 30i(t) = 25$

b.) Calculate $\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{30t}i(t)\right]$. What do you notice about this compared to the left-hand side of your differential equation?
Solution:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{30t}i(t)\right] \overset{\text{product rule}}{=} \left(\frac{\mathrm{d}}{\mathrm{d}t}e^{30t}\right)i(t) + e^{30t}\frac{\mathrm{d}i}{\mathrm{d}t} = 30e^{30t}i(t) + e^{30t}\frac{\mathrm{d}i}{\mathrm{d}t}.$$

We notice that the derivative here is the same as the left-hand side of the differential equation multiplied with e^{30t}.

c.) Multiply your differential equation on both sides by e^{30t} and then rewrite the left-hand side as $\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{30t}i(t)\right]$.
Solution: We get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{30t}i(t)\right] = 25e^{30t}.$$

d.) Solve the differential equation by integrating and solving for $i(t)$.
Solution: Integrating yields

$$\int \frac{\mathrm{d}}{\mathrm{d}t}\left[e^{30t}i(t)\right]\mathrm{d}t = \int 25e^{30t}\mathrm{d}t.$$

Since the integral of the derivative of a function is the function itself this gives us the left-hand-side. The right-hand side comes from straightforward integration:

$$e^{30t}i(t) = \frac{25}{30}e^{30t} + C.$$

Therefore

$$i(t) = \frac{25}{30} + Ce^{-30t}.$$

(*note: we do not find a value of C here because no initial condition was given!*)

2. Compute $\int_5^{11} xe^x \mathrm{d}x$.
Solution: Let $u = x$ and $\mathrm{d}v = e^x\mathrm{d}x$. Then $\mathrm{d}u = \mathrm{d}x$ and $v = e^x$. Therefore by integration by parts:

$$\int xe^x\mathrm{d}x = xe^x - \int e^x\mathrm{d}x = xe^x - e^x.$$

Thus we may compute

$$\int xe^x \mathrm{d}x = xe^x - e^x \Big|_5^{11} = \left(11e^{11} - e^{11}\right) - \left(5e^5 - e^5\right).$$

3. Compute $\int x^2 e^x \mathrm{d}x$.

   Solution: Let $u = x^2$ and $\mathrm{d}v = e^x \mathrm{d}x$. Then $\mathrm{d}u = 2x\mathrm{d}x$ and $v = e^x$. Thus by integration by parts,

$$\int x^2 e^x \mathrm{d}x = x^2 e^x - 2\int xe^x \mathrm{d}x.$$

   To compute $\int xe^x \mathrm{d}x$ we must do integration by parts again. Let $u_2 = x$ and $\mathrm{d}v_2 = e^x \mathrm{d}x$. Then $\mathrm{d}u_2 = \mathrm{d}x$ and $v_2 = e^x$. Hence

$$\int xe^x \mathrm{d}x = xe^x - \int e^x \mathrm{d}x = xe^x - e^x + C.$$

   Therefore we have shown that

$$\int x^2 e^x \mathrm{d}x = x^2 e^x - 2\left(xe^x - e^x\right) + C_1, \quad C_1 = 2C.$$

4. Compute $\int (2x+3)e^{2x} \mathrm{d}x$.

   Solution: Let $u = 2x + 3$ and $\mathrm{d}v = e^{2x}\mathrm{d}x$. Then $\mathrm{d}u = 2\mathrm{d}x$ and $v = \frac{1}{2}e^{2x}$. Using integration by parts, we compute

$$\int (2x+3)e^{2x}\mathrm{d}x = (2x+3)\frac{e^{2x}}{2} - \frac{2}{2}\int e^{2x}\mathrm{d}x = \frac{(2x+3)e^{2x}}{2} - \frac{1}{2}e^{2x} + C.$$

5. Compute $\int_2^3 \ln(x)\mathrm{d}x$.

   Solution: We know already that $\int \ln(x)\mathrm{d}x = x\ln(x) - x + C$. Use that to compute

$$\int_2^3 \ln(x)\mathrm{d}x = x\ln(x) - x\Big|_2^3 = (3\ln(3) - 3) - (2\ln(2) - 2).$$

6. Let $f(t) = e^{7t}$.

   (a) What integral must you solve in order to calculate $\mathscr{L}\{f\}(x)$ (for $x > 7$)?

   Solution: We must calculate

$$\mathscr{L}\{f\}(x) = \int_0^\infty e^{7t}e^{-xt}\mathrm{d}t.$$

(b) Calculate $\mathscr{L}\{f\}(x)$ as an improper integral.
Solution: Calculate for $x > 7$,

$$\begin{aligned}
\mathscr{L}\{f\}(x) &= \int_0^\infty e^{7t}e^{-xt}\mathrm{d}t \\
&= \lim_{b\to\infty}\int_0^b e^{(7-x)t}\mathrm{d}t \\
&\overset{u=(7-x)t,\frac{1}{7-x}\mathrm{d}u=\mathrm{d}t}{=} \lim_{b\to\infty}\frac{1}{7-x}\int_{t=0}^{t=b} e^u\mathrm{d}u \\
&= \frac{1}{7-x}\lim_{b\to\infty} e^{(7-x)t}\Big|_0^b \\
&= \frac{1}{7-x}\lim_{b\to\infty}\left[e^{(7-x)b}-1\right] \\
&= \frac{-1}{7-x} \\
&= \frac{1}{x-7}.
\end{aligned}$$

Note that the limit of $e^{(7-x)b}$ tends to zero since whenever $x > 7$ we have $7 - x < 0$.

7. Let $f(t) = t$.

(a) Let $x > 0$. What integral must you solve in order to calculate $\mathscr{L}\{f\}(x)$?
Solution: $\int_0^\infty te^{-xt}\mathrm{d}t$

(b) Calculate $\mathscr{L}\{f\}(x)$ using integration by parts and improper integration.
Solution: Let $u = t$ and $\mathrm{d}v = e^{-xt}$. Then $\mathrm{d}u = \mathrm{d}t$ and $v = -\frac{1}{x}e^{-xt}$.
So,

$$\int te^{-xt}\mathrm{d}t = -\frac{t}{x}e^{-xt} - \int\left(-\frac{1}{x}\right)e^{-xt}\mathrm{d}t = \frac{-te^{-xt}}{x} - \frac{e^{-xt}}{x^2}$$

Therefore we see that

$$\begin{aligned}
\int_0^\infty te^{-xt}\mathrm{d}t &= \lim_{b\to\infty}\int_0^b te^{-xt}\mathrm{d}t \\
&= \lim_{b\to\infty} -te^{-xt} - \frac{e^{-xt}}{x^2}\Big|_0^b \\
&= \lim_{b\to\infty}\left(-be^{-xb} - \frac{e^{-xb}}{x^2}\right) - \left(0 - \frac{1}{x^2}\right) \\
&= \frac{1}{x^2}.
\end{aligned}$$