MATH 1586 - EXAM 3 - SPRING 2018
SOLUTION

20 April 2018
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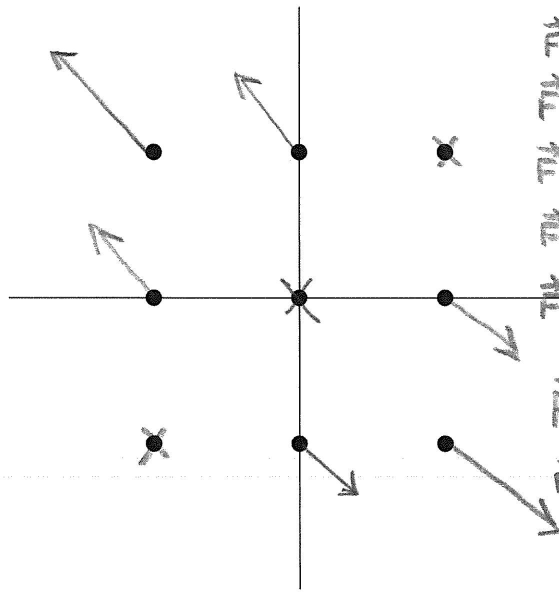
Instructions:

- Show all work, clearly and in order, if you want to get full credit. If you claim something is true **you must show work backing up your claim**. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers algebraically whenever possible to ensure full credit.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point.
- Good luck!

1. (8 points) Sketch the vector field $\vec{F}(x, y) = \langle x - y, -x + y \rangle$ on the following grid comprised of points whose x- and y- coordinates are 1, 0, or -1:



$\vec{F}(1,1) = \langle 0,0 \rangle$

$\vec{F}(-1,-1) = \langle 0,0 \rangle$

$\vec{F}(0,1) = \langle -1, 1 \rangle$

$\vec{F}(0,-1) = \langle 1, -1 \rangle$

$\vec{F}(1,0) = \langle 1, -1 \rangle$

$\vec{F}(-1,0) = \langle -1, 1 \rangle$

$\vec{F}(-1,1) = \langle -2, 2 \rangle$

$\vec{F}(1,-1) = \langle 2, -2 \rangle$

2. (12 points) Find an equation for the tangent plane to $f(x, y) = x^2y^2 + xy^2$ above the point $(-1, 2)$.

$\frac{\partial f}{\partial x} = 2xy^2 + y^2$ $\qquad$ $\frac{\partial f}{\partial y} = 2x^2y + 2xy$ $\qquad$ $f(-1,2) = 4 - 4 = 0$

$\downarrow$ at the point $(-1,2)$ $\qquad$ $\downarrow$

$\left.\frac{\partial f}{\partial x}\right|_{(-1,2)} = -2(2^2) + 2^2 = -4$ $\qquad$ $\left.\frac{\partial f}{\partial y}\right|_{(-1,2)} = 4 - 4 = 0$

Thus tangent plane is

$$z - 0 = -4(x - (-1))$$

$$\Rightarrow \boxed{z = -4(x+1)}$$

3. (12 points) Calculate the gradient of $f(x,y) = e^{xy^2}\log(x)$.

$$\nabla f = \left\langle y^2 e^{xy^2}\log(x) + \frac{e^{xy^2}}{x},\ 2ye^{xy^2}\log(x)\right\rangle$$

4. (13 points) Find the directional derivative of $f(x,y) = xy + e^{xy}$ in the direction $\langle 3,2\rangle$ at $(x_0,y_0) = (-1,-1)$.

$$\nabla f = \langle y + ye^{xy},\ x + xe^{xy}\rangle$$

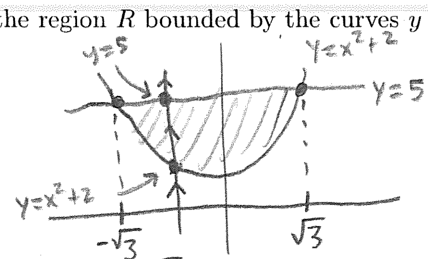
$$\nabla f(-1,-1) = \langle -1-e^{-1}, -1-e^{-1}\rangle$$

$$\vec{u} = \frac{\langle 3,2\rangle}{\|\langle 3,2\rangle\|} = \frac{\langle 3,2\rangle}{\sqrt{3^2+2^2}} = \left\langle \frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}}\right\rangle$$

So,

$$D_{\vec{u}}f(-1,-1) = \nabla f(-1,-1)\cdot\vec{u} = \frac{3}{\sqrt{13}}\left(-1-\frac{1}{e}\right) + \frac{2}{\sqrt{13}}\left(-1-\frac{1}{e}\right) = \frac{5}{\sqrt{13}}\left(-1-\frac{1}{e}\right)$$

5. (16 points) Consider the region R bounded by the curves $y = x^2+2$ and $y = 5$. Draw this region and compute $\iint_R xy\,dA$.



intersection

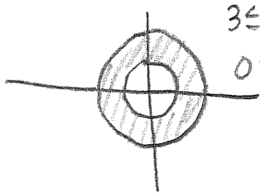
$$x^2+2 = 5$$
$$\downarrow$$
$$x^2 = 3$$
$$\downarrow$$
$$x = \pm\sqrt{3}$$

$$\int_{-\sqrt{3}}^{\sqrt{3}}\int_{x^2+2}^{5} xy\,dy\,dx = \int_{-\sqrt{3}}^{\sqrt{3}} \underbrace{\frac{xy^2}{2}\Big|_{y=x^2+2}^{y=5}}_{=\frac{25}{2}x - \frac{(x^2+2)^2x}{2}} dx = \int_{-\sqrt{3}}^{\sqrt{3}} \frac{25}{2}x - \frac{1}{2}(x^5+4x^3+4x)\,dx$$

$$= \frac{25}{4}x^2 - \frac{1}{12}x^6 + x^4 + 2x^2\Big|_{-\sqrt{3}}^{\sqrt{3}}$$

$$= 0$$

6. (13 points) Let R be the annular region between the circles of radius 3 and radius 5. Draw this region and set up **but do not evaluate** the integral to compute $\iint_R x^2 + y^2 \mathrm{d}A$.



$3 \le r \le 5$

$0 \le \theta \le 2\pi$

$$\iint_R x^2+y^2\, dA = \int_0^{2\pi}\int_3^5 r^3\, dr\, d\theta$$

7. (13 points) Set up **but do not evalute** the line integral $\int_C xy \mathrm{d}s$ where C is the curve from the point $(2,2)$ to the point $(5,7)$.

$\vec{r}(t) = (1-t)\langle 2,2\rangle + t\langle 5,7\rangle$

$= \langle 3t+2, 5t+2\rangle$

$0 \le t \le 1$

$\vec{r}\,'(t) = \langle 3,5\rangle$

$\|\vec{r}\,'(t)\| = \sqrt{9+25} = \sqrt{34}$

$$\Rightarrow \int_C xy\, ds = \int_0^1 (3t+2)(5t+2)\sqrt{34}\, dt$$

8. (13 points) Set up **but do not evaluate** the line integral $\int_C \langle x^2y^5, \log(xy)e^y\rangle \mathrm{d}\vec{r}$ where $\vec{r}(t)$ is the arc of the parabola $y = x^2$ from $(1,1)$ to $(3,9)$.

$\vec{r}(t) = \langle t, t^2\rangle$

$1 \le t \le 3$

$\vec{r}\,'(t) = \langle 1, 2t\rangle$

$$\int_C \langle x^2y^5, \log(xy)e^y\rangle\, d\vec{r} = \int_1^3 \langle t^2 \cdot t^{10}, \log(t^3)e^{t^2}\rangle \cdot \langle 1, 2t\rangle\, dt$$

$$= \int_1^3 t^{12} + 2te^{t^2}\log(t^3)\, dt$$